

Abstract Concept of Changeable Set

Grushka Ya.I.

*Institute of Mathematics NAS of Ukraine.
3, Tereshchenkivska st.,
Kyiv(Kiev)-4,
01601 Ukraine
e-mail: grushka@imath.kiev.ua*

Abstract. The work lays the foundations of the theory of changeable sets. In author opinion, this theory, in the process of it's development and improvement, can become one of the tools of solving the sixth Hilbert problem least for physics of macrocosm. From a formal point of view, changeable sets are sets of objects which, unlike the elements of ordinary (static) sets may be in the process of continuous transformations, and which may change properties depending on the point of view on them (the area of observation or reference frame). From the philosophical and intuitive point of view the changeable sets can look like as "worlds" in which changes obey arbitrary laws.

Key words: changeable sets, movement, evolution, sixth Hilbert's problem

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1 Introduction

In spite of huge success of modern theoretical physics and the power of mathematical tools, which it applies, its foundations remain unclear. Well-known sixth Hilbert's problem of mathematically strict formulation of the foundations of theoretical physics, delivered in 1900 [1], completely is not solved to this day [2, 3]. Some attempts to formalize certain physical theories was done in the papers [4-9]. The main defect of these works is the absence of a single abstract and systematic approach, and, consequently, insufficiency of flexibility of the mathematical apparatus of these works, excessive its adaptability to the specific physical theories under consideration. Moreover trying in [6] to immediately formalize the maximum number of known physical objects, without creating a hierarchy of elementary abstract mathematical concepts has led to not very easy for the analysis mathematical object [6, page. 177, definition 4.1]. In general, it should be noted, that the main feature of existing mathematically strict models of theoretical physics is that the investigators try to find intuitively the mathematical tools to describe physical phenomena under consideration, and only then they try to formalize the description of this phenomena, identifying physical objects with some constructs, generated by these mathematical tools, for example, with solutions of some differential equations on some space or manifold. As a result, quite complicated mathematical structures appear, whereas most elementary physical concepts and postulates, obtained by a help of experiments, life experience or common sense (which led to the appearance of this mathematical model), remain not formulated mathematically strictly. In works [10, 11] it is expressed the view that, in the general case, it is impossible to solve this problem by means of existing mathematical theories. Also in [10, 11] it is posed the problem of constructing the theory of "dynamic sets", that is the theory of new abstract mathematical structures for modeling various processes in physical, biological and other complex systems.

In the present work the foundations of the theory of changeable sets are laid and the basic properties of these sets are established. The theory of changeable sets can be considered as attempt to give a solution of the problem, posed in [10, 11]. And author hopes, that the apparatus of the theory of changeable sets can generate the necessary mathematical structures at least for physics and some other natural sciences in macrocosm.

From a formal point of view, changeable sets are sets of objects which, unlike the elements of ordinary (static) sets may be in the process of continuous transformations, and which may change properties depending on the point of view on them (the area of observation or reference frame). From the philosophical and intuitive point of view the changeable sets can look like as "worlds" in which changes obey arbitrary laws. Note that the main statements of the theory of changeable sets has been announced in [12]. Fundamentals of the theory of primitive changeable sets (which is contained in the sections 2–6 of this paper) also has been presented in [17].

2 Oriented Sets and their Properties

When we try to see on any picture of reality (area of reality) from the most abstract point of view, we can only say that this picture in every moment of its existence consists of certain things (objects). During the research of this area of reality the objects of which it consists can be divided into smaller, elementary, objects that we call the elementary states. Method of division a given area of reality into elementary states depends on our knowledge of this area, of practice required level of detailing of research, or of the level of physical and mathematical idealization of the studied system. Depending on these factors as elementary states may be, for example, the position of a material point or an elementary particle in a given time, the value of scalar, vector or tensor field at a given point in space-time, state of individuals of a species in a given time (in mathematical models of biology) and others. And if a picture of reality will not changed over time, then to describe this picture of reality (in the most

abstract form) it is sufficient the classical set theory, when elementary states are interpreted as elements of a certain set. However, the reality is changeable. Elementary states in the process of evolution can change their properties (and thus lose its formal mathematical self-identity), also elementary states may born or disappear, decompose into several elementary states, or, conversely, several elementary states may merge into one. But it is obvious that whenever it is possible to trace the “evolution lines” of the studied system, we can clearly answer the question whether the elementary state “ y ” is the result of transformations (ie, “transformation descendant”) of elementary state “ x ”. Therefore, the next definition may be considered as the simplest (starting) model of a set of changing objects.

Definition 2.1. *Let, M be any NON-EMPTY set.*

*Arbitrary reflexive binary relation \Leftarrow on M (that is a relation satisfying $\forall x \in M \ x \Leftarrow x$) we will name an **orientation**, and the pair $\mathcal{M} = (M, \Leftarrow)$ will be called an **oriented set**. In this case the set M we will name a **basic set** or a set of all **elementary states** of oriented set \mathcal{M} and we will denote it by $\mathfrak{Bs}(\mathcal{M})$. The relation \Leftarrow we will name a **directing relation of changes (transformations)** of \mathcal{M} , and we will denote it by $\xleftarrow{\mathcal{M}}$.*

In the case where the oriented set \mathcal{M} is known in advance, the char \mathcal{M} in the denotation $\xleftarrow{\mathcal{M}}$ will be released, and we will use denotation \leftarrow instead. For the elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ the record $y \leftarrow x$ should be read as “the elementary state y is the result of transformations (or the transformation prolongation) of the elementary state x ”

Remark 2.1. Some attempts to construct abstract mathematical structures for modeling physical systems made in [8,9]. In these works as a basic abstract model it is proposed to consider a pair of kind (M, \prec) , where M is some set and \prec is the local sequence relation (that is asymmetric and locally transitive (in the sense of [9, page 28]) relation), which satisfies the additional axioms TK₁-TK₃ [9]. The main deficiency of this approach is, that it is not motivated by abstract philosophical arguments, while the main motivation is provided by the specific example of order relation, generated by the “light cone” in Minkowski space-time. Due to these factors, the model, suggested in [8,9], is not enough flexible. In particular, this model is unusable for the description of discrete processes. Also, this model is not enough comfortable for consideration (at the abstract level) of complex branched processes, where different “branches” of the process can “intersect” or “merge” during transformations. Moreover the construction of mathematical model of the special relativity theory, based on the order relation of “light cone” makes it impossible the mathematically strict study of tachyons under this model.

Note, that the directing relation of changes in the definition 2.1 displays only real transformations, of the elementary states which have appeared in the oriented set, while the the local sequence relation [8,9] (in particular “light cone” order relation), display all potentially possible transformations.

Let \mathcal{M} be an oriented set.

Definition 2.2. *The subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$ will be referred to as **transitive** in \mathcal{M} if for any $x, y, z \in N$ such, that $z \leftarrow y$ and $y \leftarrow x$ we have $z \leftarrow x$.*

*The transitive subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$ will be called **maximum transitive** if there not exist a transitive set $N_1 \subseteq \mathfrak{Bs}(\mathcal{M})$, such, that $N \subset N_1$ (where the symbol \subset denotes the strict inclusion, that is $N \neq N_1$).*

*The transitive subset $L \subseteq \mathfrak{Bs}(\mathcal{M})$ will be referred to as **chain** in \mathcal{M} if for any $x, y \in L$ at least one of the relations $y \leftarrow x$ or $x \leftarrow y$ is true. The chain $L \subseteq \mathfrak{Bs}(\mathcal{M})$ we will name the **maximum chain** if there not exist a chain $L_1 \subseteq \mathfrak{Bs}(\mathcal{M})$, such, that $L \subset L_1$.*

Assertion 2.1. *Let \mathcal{M} be an oriented set.*

1. *Any non-empty subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$, containing not more than, two elements is transitive.*

2. Any non-empty subset $L = \{x, y\} \subseteq \mathfrak{Bs}(\mathcal{M})$, containing not more than, two elements is chain if and only if $y \leftarrow x$ or $x \leftarrow y$. In particular, any singleton $L = \{x\} \subseteq \mathfrak{Bs}(\mathcal{M})$ is chain.

The proof of the assertion 2.1 is reduced to trivial verification.

Lemma 2.1. *Let \mathcal{M} be an oriented set.*

1. *Union of an arbitrary family of transitive sets of \mathcal{M} , linearly ordered by the inclusion relation, is a transitive set in \mathcal{M} .*
2. *Union of an arbitrary family of chains of \mathcal{M} , linearly ordered by the inclusion relation, is a chain in \mathcal{M} .*

Proof. 1. Let $\mathfrak{N} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be a family of transitive sets of \mathcal{M} , linearly ordered by the inclusion relation. Denote:

$$\tilde{N} := \bigcup_{N \in \mathfrak{N}} N.$$

Consider any elementary states $x, y, z \in \tilde{N}$ such, that $z \leftarrow y$ and $y \leftarrow x$. Since $x, y, z \in \tilde{N} = \bigcup_{N \in \mathfrak{N}} N$ then there exist $N_x, N_y, N_z \in \mathfrak{N}$ such, that $x \in N_x, y \in N_y, z \in N_z$. Since the family of sets \mathfrak{N} is linearly ordered by the inclusion relation, then there exists the set $N_0 \in \{N_x, N_y, N_z\}$ such, that $N_x, N_y, N_z \subseteq N_0$. So, we have $x, y, z \in N_0$. Since $N_0 \in \{N_x, N_y, N_z\} \subseteq \mathfrak{N}$, then N_0 is the transitive set. Therefore from conditions $z \leftarrow y$ and $y \leftarrow x$ it follows, that $z \leftarrow x$. Thus \tilde{N} is the transitive set.

2. Let $\mathfrak{L} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be a family of chains of \mathcal{M} , linearly ordered by the inclusion relation. Denote:

$$\tilde{L} := \bigcup_{L \in \mathfrak{L}} L.$$

By the post 1, \tilde{L} is the transitive set. Consider any elementary states $x, y \in \tilde{L}$. Since the family of sets \mathfrak{L} is linearly ordered by the inclusion relation, then, similarly as in the post 1, there exists a chain $L_0 \in \mathfrak{L}$ such, that $x, y \in L_0$. And, because L_0 is chain, at least one of the relations $y \leftarrow x$ or $x \leftarrow y$ is true. Thus \tilde{L} is the chain of \mathcal{M} . \square

Using the lemma 2.1 and the Zorn's lemma, we obtain the following assertion.

Assertion 2.2.

1. *For any transitive set N of oriented set \mathcal{M} there exists a maximum transitive set N_{\max} such, that $N \subseteq N_{\max}$.*
2. *For any chain L of oriented set \mathcal{M} there exists a maximum chain L_{\max} such, that $L \subseteq L_{\max}$.*

It should be noted that the second post of the assertion 2.2 can be referred to as the generalization of the Hausdorff maximal principle under this theory.

The following corollaries results from the assertions 2.2 and 2.1.

Corollary 2.1. *For any two elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ in the oriented set \mathcal{M} there exists a maximum transitive set $N \subseteq \mathfrak{Bs}(\mathcal{M})$ such that $x, y \in N$.*

Corollary 2.2. *For any two elements $x, y \in \mathfrak{Bs}(\mathcal{M})$, such that $y \leftarrow x$, in the oriented set \mathcal{M} there exists a maximum chain L such that $x, y \in L$.*

If we put $x = y \in \mathfrak{Bs}(\mathcal{M})$ (by definition $\mathfrak{Bs}(\mathcal{M}) \neq \emptyset$), we obtain, that in any oriented set \mathcal{M} necessarily there exist maximum transitive sets and maximum chains.

3 Definition of the Time. Primitive Changeable Sets

In theoretical physics, scientists tend to think, that the moments of time are real numbers. But the abstract mathematics deal with objects of a arbitrarily large cardinality. Therefore in out abstract theory we will not restricted to the real moments of time. In the next definition moments of time are elements of any linearly ordered set. Such definition of time is close to the philosophy conception of time as some “chronological order”, agreed upon the processes of transformations.

Definition 3.1. Let \mathcal{M} be an oriented set and $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set. A map $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ is referred to as **time** on \mathcal{M} iff the following conditions are satisfied:

- 1) For any elementary state $x \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ there exists a element $t \in \mathbf{T}$ such that $x \in \psi(t)$.
- 2) If $x_1, x_2 \in \mathcal{M}$, $x_2 \leftarrow x_1$ and $x_1 \neq x_2$, then there exist elements $t_1, t_2 \in \mathbf{T}$ such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$ (this means that there is a temporal separateness of successive unequal elementary states).

In this case the elements $t \in \mathbf{T}$ we will call the moments of time, the pair

$$\mathcal{H} = (\mathbb{T}, \psi) = ((\mathbf{T}, \leq), \psi)$$

will be named **chronologization** of \mathcal{M} and the triple

$$\mathcal{P} = (\mathcal{M}, \mathbb{T}, \psi) = (\mathcal{M}, (\mathbf{T}, \leq), \psi)$$

we will call the **primitive changeable set**.

Remark 3.1. In [8,9] linearly ordered sets has been used as time-scales also. But the conception of time in the definition 3.1 is significantly different from [8,9]. Note, that the definition of time in [8,9] is less general, then the definition 3.1 due to less generality of the model, suggested in [8,9].

We say that an oriented set **can be chronologized** if there exists at least one chronologization of \mathcal{M} . It turns out that any oriented set can be chronologized. To make sure this we may consider any linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, which contains at least two elements and put:

$$\psi(t) := \mathfrak{B}\mathfrak{s}(\mathcal{M}), \quad t \in \mathbf{T}.$$

The conditions of the definition 3.1 for the function $\psi(\cdot)$ apparently are satisfied. More non-trivial methods to chronologize an oriented set we will consider in the section 4.

The following two assertions (3.1 and 3.2) are trivial consequences of the definition 3.1.

Assertion 3.1. Let \mathcal{M} and \mathcal{M}_1 be oriented sets, and while $\mathfrak{B}\mathfrak{s}(\mathcal{M}) \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{M}_1)$ and $\leftarrow_{\mathcal{M}} \subseteq \leftarrow_{\mathcal{M}_1}$ (last inclusion means that for $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ the condition $y \leftarrow_{\mathcal{M}} x$ implies $y \leftarrow_{\mathcal{M}_1} x$).

If a mapping $\psi_1 : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M}_1)}$ (where $\mathbb{T} = (\mathbf{T}, \leq)$ is a linearly ordered set) is a time on \mathcal{M}_1 then the mapping:

$$\psi(t) = \psi_1(t) \cap \mathfrak{B}\mathfrak{s}(\mathcal{M})$$

is the time on \mathcal{M} .

Assertion 3.2. Let \mathcal{M} be an oriented set and $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ be a time on \mathcal{M} .

(1) If $\mathbf{T}_1 \subseteq \mathbf{T}$, $\mathbf{T}_1 \neq \emptyset$ and $\psi(t) = \emptyset$ for $t \in \mathbf{T} \setminus \mathbf{T}_1$, then the mapping $\psi_1 = \psi \upharpoonright \mathbf{T}_1$, which is the restriction of ψ on the set \mathbf{T}_1 also is time on \mathcal{M} .

(2) If the ordered set \mathbf{T} is embedded in a linearly ordered set $(\tilde{\mathbf{T}}, \leq_1)$ (preserving order), then the mapping $\tilde{\psi} : \tilde{\mathbf{T}} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$:

$$\tilde{\psi}(t) = \begin{cases} \psi(t), & t \in \mathbf{T} \\ \emptyset, & t \in \tilde{\mathbf{T}} \setminus \mathbf{T} \end{cases}$$

also is time on \mathcal{M} .

The assertion 3.2 affirms, that “moments of full death” may be erased from or added to “chronological history” of primitive changeable set.

4 One-point and Monotone Time. Chronologization Theorems

Definition 4.1. Let $(\mathcal{M}, \mathbb{T}, \psi) = (\mathcal{M}, (\mathbb{T}, \leq), \psi)$ be a primitive changeable set.

- 1) The time ψ will be called **quasi one-point** if for any $t \in \mathbb{T}$ the set $\psi(t)$ is a singleton.
- 2) The time ψ will be called **one-point** if the following conditions are satisfied:
 - (a) The time ψ is quasi one-point;
 - (b) If $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$ then $x_2 \leftarrow x_1$.
- 3) The time ψ will be called **monotone** if for any elementary states $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ the conditions $x_2 \leftarrow x_1$ and $x_1 \not\leftarrow x_2$ imply $t_1 < t_2$.

In the case, when the time ψ is quasi one-point (one-point/monotone) the chronologization (\mathbb{T}, ψ) of the oriented set \mathcal{M} will be called quasi one-point (one-point/monotone) correspondingly.

Example 4.1. Let us consider an arbitrary mapping $f : \mathbb{R} \mapsto \mathbb{R}^d$ ($d \in \mathbb{N}$). This mapping can be interpreted as equation of motion of single material point in the space \mathbb{R}^d . This mapping generates the oriented set $\mathcal{M} = \left(\mathfrak{Bs}(\mathcal{M}), \overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}} \right)$, where $\mathfrak{Bs}(\mathcal{M}) = \mathfrak{R}(f) = \{f(t) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^d$ and for $x, y \in \mathfrak{Bs}(\mathcal{M})$ the correlation $y \overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}} x$ is true if and only if there exist $t_1, t_2 \in \mathbb{R}$ such, that $x = f(t_1)$, $y = f(t_2)$ and $t_1 \leq t_2$. It is easy to verify, that the mapping:

$$\psi(t) = \{f(t)\} \subseteq \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathbb{R}.$$

is a one-point time on \mathcal{M} .

The example 4.1 makes clear the definition of one-point time. It is evident, that *any one-point time is quasi one-point and monotone*. It turns out that a quasi one-point time need not be monotone (and thus one-point), and monotone time need not be quasi one-point (and thus one-point). The next examples prove what is written above.

Example 4.2. Let us consider any two element set $M = \{x_1, x_2\}$. We construct the oriented set $\mathcal{M} = \left(\mathfrak{Bs}(\mathcal{M}), \overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}} \right)$ by the following way:

$$\begin{aligned} \mathfrak{Bs}(\mathcal{M}) &= M = \{x_1, x_2\}; \\ \overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}} &= \{(x_2, x_1), (x_1, x_1), (x_2, x_2)\} \end{aligned}$$

(or, in other words, $x_2 \overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}} x_1$, $x_1 \overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}} x_1$, $x_2 \overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}} x_2$). Note that the directing relation of changes $\overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}}$ can be represented in more laconic form: $\overset{\leftarrow}{\underset{\mathcal{M}}{\leftarrow}} = \{(x_2, x_1)\} \cup \text{diag}(M)$, where $\text{diag}(M) = \{(x, x) \mid x \in M\}$. As a linearly ordered set we take $\mathbb{T} = (\mathbb{R}, \leq)$ (with the usual order on the real field). On the oriented set \mathcal{M} we define time by the following way:

$$\psi(t) := \begin{cases} \{x_1\}, & t \notin \mathbb{Q}; \\ \{x_2\}, & t \in \mathbb{Q}, \end{cases}$$

where \mathbb{Q} is the field of rational numbers. It is easy to verify, that the mapping ψ really is time on \mathcal{M} in the sense of definition 3.1. Since $\psi(t)$ is a singleton for any $t \in \mathbb{R}$, the time ψ is quasi one-point. If we put $t_1 = \sqrt{2}$, $t_2 = 1$, we obtain $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$, $x_2 \leftarrow x_1$, $x_1 \not\leftarrow x_2$, but $t_1 > t_2$. Thus time ψ is not monotone.

Example 4.3. Let us consider an arbitrary four-element set $M = \{x_1, x_2, x_3, x_4\}$ and construct the oriented set $\mathcal{M} = (\mathfrak{Bs}(\mathcal{M}), \leftarrow_{\mathcal{M}})$ by the following way:

$$\begin{aligned}\mathfrak{Bs}(\mathcal{M}) &= M = \{x_1, x_2, x_3, x_4\}; \\ \leftarrow_{\mathcal{M}} &= \{(x_2, x_1), (x_4, x_3)\} \cup \text{diag}(M).\end{aligned}$$

As a linearly ordered set we take $\mathbb{T} = (\{1, 2\}, \leq)$ (with the usual ordering on the real axis). Time on \mathcal{M} is defined by the following way:

$$\psi(t) := \begin{cases} \{x_1, x_3\}, & t = 1 \\ \{x_2, x_4\}, & t = 2. \end{cases}$$

It is not hard to prove, that the mapping ψ is monotone time on \mathcal{M} . But this time, obviously, is not quasi one-point.

It appears that quasi one-point and, simultaneously, monotone time need not be one-point. This fact is illustrated by the following example.

Example 4.4. Let the oriented set \mathcal{M} be same as in the example 4.3. We consider the ordered set $\mathbb{T} = (\{1, 2, 3, 4\}, \leq)$ (with the usual real number ordering). Time on \mathcal{M} we define by the following:

$$\psi(t) := \{x_t\}, \quad t \in \{1, 2, 3, 4\}.$$

It is not hard to verify, that $\psi(\cdot)$ is quasi one-point and monotone time on \mathcal{M} . Although, if we put $t_1 := 2$, $t_2 := 3$, we receive, $x_2 \in \psi(t_1)$, $x_3 \in \psi(t_2)$, $t_1 \leq t_2$, but $x_3 \not\leftarrow x_2$. Thus, the time ψ is not one-point.

Definition 4.2. Oriented set \mathcal{M} will be called a **chain oriented set** if the set $\mathfrak{Bs}(\mathcal{M})$ is the chain of \mathcal{M} , that is if the relation \leftarrow is transitive on $\mathfrak{Bs}(\mathcal{M})$ and for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ at least one of the conditions $x \leftarrow y$ or $y \leftarrow x$ is satisfied.

Oriented set \mathcal{M} will be called a **cyclic** if for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ both of the relations $x \leftarrow y$ and $y \leftarrow x$ are true.

It is evident, that any cyclic oriented set is a chain.

Lemma 4.1. Any cyclic oriented set can be one-point chronologized.

Proof. Let \mathcal{M} be a cyclic oriented set. By definition of oriented set, $\mathfrak{Bs}(\mathcal{M}) \neq \emptyset$. Choose any two disjoint sets $\mathbf{T}_1, \mathbf{T}_2$ equipotent to the set $\mathfrak{Bs}(\mathcal{M})$ ($\mathbf{T}_1 \cap \mathbf{T}_2 = \emptyset$). (Such sets must exist, because we can put $\mathbf{T}_1 := \mathfrak{Bs}(\mathcal{M})$ and construct the set \mathbf{T}_2 from the elements of set $\tilde{\mathbf{T}} = 2^{\mathbf{T}_1} \setminus \mathbf{T}_1$, cardinality of which is not smaller the cardinality of \mathbf{T}_1 .) Let \leq_i ($i = 1, 2$) be any linear order relation on \mathbf{T}_i (by Zermelo's theorem, such linear order relations necessarily exist). Denote:

$$\mathbf{T} := \mathbf{T}_1 \cup \mathbf{T}_2.$$

On the set \mathbf{T} we construct the relation:

$$\leq = \leq_1 \cup \Delta_2 \cup \{(t, \tau) \mid t \in \mathbf{T}_1, \tau \in \mathbf{T}_2\},$$

or, in the other words, for $t, \tau \in \mathbf{T}$ relation $t \leq \tau$ holds if and only if one of the following conditions is satisfied:

- (O1) $t, \tau \in \mathbf{T}_i$ and $t \leq_i \tau$ for some $i \in \{1, 2\}$;
- (O2) $t \in \mathbf{T}_1, \tau \in \mathbf{T}_2$.

The pair (\mathbf{T}, \leq) is the ordered union of the linear ordered sets (\mathbf{T}_1, \leq_1) and (\mathbf{T}_2, \leq_2) . Thus, by [13, p. 208], (\mathbf{T}, \leq) is a linear ordered set. Let $f : \mathbf{T}_2 \rightarrow \mathbf{T}_1$ be any bijection (one-to-one

correspondence) between the (equipotent) sets \mathbf{T}_1 and \mathbf{T}_2 . And let $g : \mathbf{T}_1 \mapsto \mathfrak{B}\mathfrak{s}(\mathcal{M})$ be any bijection between the (equipotent) sets \mathbf{T}_1 and $\mathfrak{B}\mathfrak{s}(\mathcal{M})$.

Let us consider the following mapping $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$:

$$\psi(t) := \begin{cases} \{g(t)\}, & t \in \mathbf{T}_1; \\ \{g(f(t))\}, & t \in \mathbf{T}_2. \end{cases} \quad (1)$$

We are going to prove, that ψ is a time on the oriented set \mathcal{M} .

1) Let $x \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$. Since the mapping $g : \mathbf{T}_1 \mapsto \mathfrak{B}\mathfrak{s}(\mathcal{M})$ is bijection between the sets \mathbf{T}_1 and $\mathfrak{B}\mathfrak{s}(\mathcal{M})$, there exists the inverse mapping $g^{[-1]} : \mathfrak{B}\mathfrak{s}(\mathcal{M}) \mapsto \mathbf{T}_1$. Let us consider the element $t_x = g^{[-1]}(x) \in \mathbf{T}_1 \subseteq \mathbf{T}$. According to (1):

$$\psi(t_x) = \{g(t_x)\} = \{g(g^{[-1]}(x))\} = \{x\}.$$

Therefore, $x \in \psi(t_x)$. Thus the first condition of the time definition 3.1 is satisfied.

2) Let $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ be elements of $\mathfrak{B}\mathfrak{s}(\mathcal{M})$ such, that $y \leftarrow x$ and $x \neq y$. Denote:

$$\begin{aligned} t_x &:= g^{[-1]}(x) \in \mathbf{T}_1; \\ t_y &:= f^{[-1]}(g^{[-1]}(y)) \in \mathbf{T}_2. \end{aligned}$$

By (O2), $t_x \leq t_y$. Since $\mathbf{T}_1 \cap \mathbf{T}_2 = \emptyset$, we have $t_x \neq t_y$. Thus $t_x < t_y$. In accordance with (1), we obtain:

$$\begin{aligned} \psi(t_x) &= \{g(t_x)\} = \{g(g^{[-1]}(x))\} = \{x\}; \\ \psi(t_y) &= \{g(f(t_y))\} = \{g(f(f^{[-1]}(g^{[-1]}(y))))\} = \{y\}. \end{aligned}$$

Consequently, $x \in \psi(t_x)$, $y \in \psi(t_y)$. That is the second condition of the time definition 3.1 also is satisfied.

Thus ψ is a time on \mathcal{M} . It remains to prove that the time ψ is one-point.

According to (1), for any $t \in \mathbf{T}$ the set $\psi(t)$ consists of one element (is a singleton). Thus the condition (a) of the one-point time definition 4.1 is satisfied. Since the oriented set \mathcal{M} is a cyclic, the condition (b) of the definition 4.1 is also satisfied. Thus time ψ is one-point. \square

Theorem 4.1. *Any chain oriented set can be one-point chronologized.*

Proof. Let \mathcal{M} be a chain oriented set. Then the set $\mathfrak{B}\mathfrak{s}(\mathcal{M})$ is a chain of oriented set \mathcal{M} , ie the relation $\leftarrow = \bigcup_{\mathcal{M}} \leftarrow$ is quasi order on $\mathfrak{B}\mathfrak{s}(\mathcal{M})$.

We will say that elements $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ are *cyclic equivalent* (denotation $x \overset{\circ}{\equiv} y$) if $x \leftarrow y$ and $y \leftarrow x$. In accordance with [14, page 21], relation $\overset{\circ}{\equiv}$ is equivalence relation on $\mathfrak{B}\mathfrak{s}(\mathcal{M})$. Let F_1 and F_2 be any two classes of equivalence, generated by the relation $\overset{\circ}{\equiv}$. We will denote $F_2 \leftarrow F_1$ if for any $x_1 \in F_1$, $x_2 \in F_2$ it is true $x_2 \leftarrow x_1$. According to [14, page 21], the relation \leftarrow is ordering on the quotient set $\mathfrak{B}\mathfrak{s}(\mathcal{M}) / \overset{\circ}{\equiv}$ of all equivalence classes, generated by $\overset{\circ}{\equiv}$. We aim to prove, that this ordering is linear. Chose any equivalence classes $F_1, F_2 \in \mathfrak{B}\mathfrak{s}(\mathcal{M}) / \overset{\circ}{\equiv}$. Because F_1, F_2 are equivalence classes, they are nonempty, therefore there exists the elements $x_1 \in F_1$, $x_2 \in F_2$. Since the oriented set \mathcal{M} is a chain, at least one from the relations $x_2 \leftarrow x_1$ or $x_1 \leftarrow x_2$ is true. But, because any two elements, belonging to the same class of equivalence, are cyclic equivalent, in the case $x_2 \leftarrow x_1$ we will have $F_2 \leftarrow F_1$, and in the case $x_1 \leftarrow x_2$ we obtain $F_1 \leftarrow F_2$. Thus $(\mathfrak{B}\mathfrak{s}(\mathcal{M}) / \overset{\circ}{\equiv}, \leftarrow)$ is a linear ordered set.

Any equivalence class $F \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv}$ is a cyclic oriented set relatively the relation \leftarrow (restricted to this class). Consequently, by lemma 4.1, any such equivalence class can be one-point chronologized. Let $(\mathbf{T}_F, \psi_F) = ((\mathbf{T}_F, \leq_F), \psi_F)$ be a one-point chronologization of the class of equivalence $F \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv}$. Without loss of generality we can assume that $\mathbf{T}_F \cap \mathbf{T}_G = \emptyset$ for $F \neq G$. Indeed, otherwise we may use the sets:

$$\tilde{\mathbf{T}}_F = \{(t, F) : t \in \mathbf{T}_F\}, \quad F \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv},$$

with ordering:

$$(t_1, F) \lesssim_F (t_2, F) \iff t_1 \leq_F t_2, \quad t_1, t_2 \in \mathbf{T}_F \quad (F \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv})$$

and times:

$$\tilde{\psi}_F((t, F)) = \psi_F(t), \quad t \in \mathbf{T}_F \quad (F \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv}),$$

it is evident, that these times are one-point.

Thus, we will assume that $\mathbf{T}_F \cap \mathbf{T}_G = \emptyset$, $F \neq G$. Denote:

$$\mathbf{T} := \bigcup_{F \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv}} \mathbf{T}_F.$$

According to this denotation, for any element $t \in \mathbf{T}$ there exists an equivalence class $F(t) \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv}$ such, that $t \in \mathbf{T}_{F(t)}$. Since $\mathbf{T}_F \cap \mathbf{T}_G = \emptyset$, $F \neq G$, such equivalence class $F(t)$ is for an element $t \in \mathbf{T}$ unique, ie the following assertion is true:

(F) For any element $t \in \mathbf{T}$ the condition $t \in \mathbf{T}_F$ ($F \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv}$) results in $F = F(t)$.

For arbitrary elements $t, \tau \in \mathbf{T}$ we will denote $t \leq \tau$ if and only if at least one of the following conditions is true:

(O1) $F(t) \neq F(\tau)$ and $F(\tau) \leftarrow F(t)$.

(O2) $F(t) = F(\tau)$ and $t \leq_{F(t)} \tau$.

The pair (\mathbf{T}, \leq) is the ordered union of the (linear ordered) family of linear ordered sets $(\mathbf{T}_F)_{F \in \mathfrak{B}\mathfrak{s}(\mathcal{M})/\overset{\circ}{\equiv}}$. Thus, by [13, p. 208], \leq is a linear ordering on \mathbf{T} .

Denote:

$$\psi(t) := \psi_{F(t)}(t), \quad t \in \mathbf{T}. \quad (2)$$

Since $\psi_{F(t)}(t) \subseteq F(t) \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{M})$, $t \in \mathbf{T}$, the mapping ψ reflects \mathbf{T} into $2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$. Now we are going to prove, that ψ is one-point time.

(a) Let $x \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$. Then there exists an equivalence class $\Phi \in M/\overset{\circ}{\equiv}$, such, that $x \in \Phi$. Since the mapping $\psi_\Phi : \mathbf{T}_\Phi \mapsto 2^\Phi$ is a time on the oriented set (Φ, \leftarrow) , there exists a time moment $t \in \mathbf{T}_\Phi$, such, that $x \in \psi_\Phi(t)$. Since $t \in \mathbf{T}_\Phi$, then by virtue of assertion (F) we have $\Phi = F(t)$. Therefore:

$$\psi(t) = \psi_{F(t)}(t) = \psi_\Phi(t) \ni x.$$

Thus, the first condition of the time definition 3.1 is satisfied.

(b) Let $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$, $y \leftarrow x$ and $y \neq x$. According to the item (a), there exist $t, \tau \in \mathbf{T}$ such, that $x \in \psi(t)$, $y \in \psi(\tau)$. And, using (2), we obtain, $x \in \psi(t) = \psi_{F(t)}(t) \subseteq F(t)$, $y \in F(\tau)$. Hence, since $y \leftarrow x$, for any $x' \in F(t)$, $y' \in F(\tau)$ (taking into account, that $x' \overset{\circ}{\equiv} x$, $y' \overset{\circ}{\equiv} y$), we obtain $y' \leftarrow x'$. This means, that $F(\tau) \leftarrow F(t)$. And, in the case $F(t) \neq F(\tau)$, using (O1), we obtain, $t \leq \tau$, so, taking into account, that $F(t) \neq F(\tau)$ causes $t \neq \tau$, we have $t < \tau$. Thus it remains to consider the case $F(t) = F(\tau)$. In this case we have $x, y \in F(t)$. And since $y \leftarrow x$, $y \neq x$ and $\psi_{F(t)}$ is a time on $(F(t), \leftarrow)$, there exist the elements $t', \tau' \in \mathbf{T}_{F(t)}$ such, that

$x \in \psi_{F(t)}(t')$, $y \in \psi_{F(t)}(\tau')$ and $t' <_{F(t)} \tau'$. Therefore, since $t', \tau' \in \mathbf{T}_{F(t)}$, using assertion (F), we obtain $x \in \psi_{F(t)}(t') = \psi_{F(t')}(\tau') = \psi(t')$ and $y \in \psi(\tau')$. Hence $x \in \psi(t')$, $y \in \psi(\tau')$, where, $t' <_{F(t)} \tau'$ (that is $t' \leq_{F(t)} \tau'$ and $t' \neq \tau'$). So, by (F) and (O2), we obtain $t' < \tau'$.

Thus ψ is a time on \mathcal{M} .

(c) It remains to prove, that the time ψ is one-point. Since for any equivalence class $G \in \mathfrak{Bs}(\mathcal{M})/\overset{\circ}{\equiv}$ the mapping ψ_G is a one-point time, by (2), set $\psi(t)$ is a singleton for any $t \in \mathbf{T}$. Thus, the first condition of the one-point time definition is satisfied.

Let $x \in \psi(t)$, $y \in \psi(\tau)$, where $t \leq \tau$. Then by (2) $x \in \psi(t) = \psi_{F(t)}(t) \subseteq F(t)$, $y \in F(\tau)$. And in the case $F(t) = F(\tau)$ the relation $y \leftarrow x$ follows from the relation $x \overset{\circ}{\equiv} y$. Concerning the case $F(t) \neq F(\tau)$, in this case, by (O1),(O2), we obtain $F(\tau) \leftarrow F(t)$, which involves $y \leftarrow x$. Thus, the second condition of the one-point time definition also is satisfied.

Therefore, the time ψ is one-point. \square

Theorem 4.2. *Any oriented set can be quasi one-point chronologized.*

Proof. 1. Let \mathcal{M} be an oriented set. Denote by \mathfrak{L} the set of all maximum chains of the \mathcal{M} . In accordance with theorem 4.1, for any chain $L \in \mathfrak{L}$ there exists an one-point chronologization $((\mathbf{T}_L, \leq_L), \psi_L)$ of the oriented set (L, \leftarrow) . Similarly to the proof of the theorem 4.1, without loss of generality we can assume, that $\mathbf{T}_L \cap \mathbf{T}_M = \emptyset$, $L \neq M$. Denote:

$$\mathbf{T} := \bigcup_{L \in \mathfrak{L}} \mathbf{T}_L. \quad (3)$$

Let \trianglelefteq be any linear order relation on \mathfrak{L} (by Zermelo's theorem, such linear order relation necessarily exists). By virtue of (3), for any element $t \in \mathbf{T}$ chain $L(t) \in \mathfrak{L}$ exists such, that $t \in \mathbf{T}_{L(t)}$. Since $\mathbf{T}_F \cap \mathbf{T}_G = \emptyset$ ($F \neq G$), this chain $L(t)$ is unique. This means, that the following assertion is true:

(L) For any element $t \in \mathbf{T}$ the condition $t \in \mathbf{T}_L$ ($L \in \mathfrak{L}$) causes $L = L(t)$.

Let $t, \tau \in \mathbf{T}$. We shall put $t \leq \tau$ if and only if one of the following conditions is satisfied:

(O1) $L(t) \neq L(\tau)$ and $L(t) \trianglelefteq L(\tau)$.

(O2) $L(t) = L(\tau)$ and $t \leq_{L(t)} \tau$.

The pair (\mathbf{T}, \leq) is the ordered union of the (linear ordered) family of linear ordered sets $(\mathbf{T}_L)_{L \in \mathfrak{L}}$. Thus, by [13, p. 208], (\mathbf{T}, \leq) is a linear ordered set. Denote:

$$\psi(t) := \psi_{L(t)}(t) \quad t \in \mathbf{T}. \quad (4)$$

Note, that $\psi(t) = \psi_{L(t)}(t) \subseteq L(t) \subseteq \mathfrak{Bs}(\mathcal{M})$, $t \in \mathbf{T}$.

2. We intend to prove, that the mapping $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ is a time.

2.a) Let $x \in \mathfrak{Bs}(\mathcal{M})$. In accordance with corollary 2.2, there exists the maximum chain $N_x \in \mathfrak{L}$ such, that $x \in N_x$. And, since the mapping $\psi_{N_x} : \mathbf{T}_{N_x} \mapsto 2^{N_x}$ is a time, there exists an element $t_x \in \mathbf{T}_{N_x} \subseteq \mathbf{T}$, such, that $x \in \psi_{N_x}(t_x)$. Since $t_x \in \mathbf{T}_{N_x}$, by assertion (L) (see above) we have $N_x = L(t_x)$. Therefore:

$$\psi(t_x) = \psi_{L(t_x)}(t_x) = \psi_{N_x}(t_x) \ni x.$$

Thus, for any element $x \in \mathfrak{Bs}(\mathcal{M})$ always an element $t_x \in \mathbf{T}$ exists, such, that $x \in \psi(t_x)$.

2.b) Let $x, y \in \mathfrak{Bs}(\mathcal{M})$, $y \leftarrow x$, $x \neq y$. According to the corollary 2.2, a maximum chain $N_{xy} \in \mathfrak{L}$ exists, such, that $x, y \in N_{xy}$. Since $y \leftarrow x$, $x \neq y$ and mapping $\psi_{N_{xy}} : \mathbf{T}_{N_{xy}} \mapsto 2^{N_{xy}}$ is a time, there exist elements $t_x, t_y \in \mathbf{T}_{N_{xy}}$ such, that $t_x <_{N_{xy}} t_y$ (ie $t_x \leq_{N_{xy}} t_y$, $t_x \neq t_y$) and $x \in \psi_{N_{xy}}(t_x)$, $y \in \psi_{N_{xy}}(t_y)$. Since $t_x, t_y \in \mathbf{T}_{N_{xy}}$, in accordance with assertion (L), we obtain $L(t_x) = L(t_y) = N_{xy}$. Therefore:

$$\psi(t_x) = \psi_{L(t_x)}(t_x) = \psi_{N_{xy}}(t_x) \ni x;$$

$$\psi(t_y) = \psi_{L(t_y)}(t_y) = \psi_{N_{xy}}(t_y) \ni y.$$

Since $L(t_x) = L(t_y) = N_{xy}$, $t_x \leq_{N_{xy}} t_y$ and $t_x \neq t_y$, by (O2) we obtain $t_x \leq t_y$ and $t_x \neq t_y$, that is $t_x < t_y$.

Consequently for any elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $y \leftarrow x$, $x \neq y$ there exists elements $t_x, t_y \in \mathbf{T}$, such, that $t_x < t_y$, $x \in \psi(t_x)$, $y \in \psi(t_y)$.

Thus, the mapping $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ really is a time on \mathcal{M} .

3. Since the times $\{\psi_L | L \in \mathfrak{L}\}$ are one point, from (4) it follows, that for any $t \in \mathbf{T}$ the set $\psi(t)$ is a singleton. Thus, the time ψ is quasi one-point. \square

It is clear, that any oriented set \mathcal{M} , containing elementary states $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_2 \not\leftarrow x_1$ and $x_1 \not\leftarrow x_2$, can not be one-point chronologized. Thus, not any oriented set can be one-point chronologized. The next assertion shows, that not any oriented set can be monotone chronologized.

Assertion 4.1. *If oriented set \mathcal{M} contains elementary states $x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_2 \leftarrow x_1$, $x_1 \not\leftarrow x_2$, $x_3 \leftarrow x_2$, $x_2 \not\leftarrow x_3$, $x_1 \leftarrow x_3$, $x_1 \neq x_3$, then this oriented set can not be monotone chronologized.*

Proof. Let oriented set \mathcal{M} contains elementary states $x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$, satisfying the conditions of assertion. Suppose, that the monotone chronologization $((\mathbf{T}, \leq), \psi)$ of the oriented set \mathcal{M} exists. This means, that the mapping $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ is a monotone time on \mathcal{M} . Since $x_1 \leftarrow x_3$ and $x_1 \neq x_3$, by time definition 3.1, there exist time points $t_1, t_3 \in \mathbf{T}$ such, that $x_1 \in \psi(t_1)$, $x_3 \in \psi(t_3)$ and $t_3 < t_1$. Also, by time definition 3.1, there exists time point $t_2 \in \mathbf{T}$, such, that $x_2 \in \psi(t_2)$. Then, by definition of monotone time 4.1, from conditions $x_2 \leftarrow x_1$, $x_1 \not\leftarrow x_2$, $x_3 \leftarrow x_2$, $x_2 \not\leftarrow x_3$ it follows, that $t_1 < t_2$, $t_2 < t_3$. Hence $t_1 < t_3$, which contradicts inequality above ($t_3 < t_1$). Thus, the assumption about the existence of monotone chronologization of \mathcal{M} is wrong. \square

Problem 4.1. *Find necessary and sufficient conditions of existence of monotone chronologization for oriented set.*

5 Time and Simultaneity. Internal Time

Definition 5.1. *Let $(\mathcal{M}, \mathbf{T}, \psi) = (\mathcal{M}, (\mathbf{T}, \leq), \psi)$ be a primitive changeable set. The set*

$$Y_\psi = \{\psi(t) \mid t \in \mathbf{T}\}$$

*will be referred to as the **set of simultaneous states**, generated by the time ψ .*

Directly from the time definition (definition 3.1) it follows the next assertion.

Assertion 5.1. *Let $(\mathcal{M}, \mathbf{T}, \psi) = (\mathcal{M}, (\mathbf{T}, \leq), \psi)$ be a primitive changeable set, and Y_ψ be a set of simultaneous states, generated by the time ψ . Then:*

$$\bigcup_{A \in Y_\psi} A = \mathfrak{Bs}(\mathcal{M}).$$

Definition 5.2. *Let \mathcal{M} be an oriented set. Any family of sets $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$, which possesses the property $\bigcup_{A \in \mathbf{Y}} A = \mathfrak{Bs}(\mathcal{M})$ we will call the **simultaneity** on \mathcal{M} .*

According to the assertion 5.1, any set of simultaneous states, generated by the time ψ of a primitive changeable set $(\mathcal{M}, \mathbf{T}, \psi)$ is a simultaneity.

Let \mathbf{Y} be a simultaneity on an oriented set \mathcal{M} and $A, B \in \mathbf{Y}$. We will denote $B \leftarrow A$ (or $B \leftarrow_{\mathcal{M}} A$) if and only if:

$$A = B = \emptyset, \text{ or } \exists x \in A \exists y \in B (y \leftarrow x).$$

The next lemma is trivial.

Lemma 5.1. *Let \mathbf{Y} be a simultaneity on an oriented set \mathcal{M} . Then the pair (\mathbf{Y}, \leftarrow) itself is an oriented set.*

Theorem 5.1. *Let \mathcal{M} be an oriented set and $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be a simultaneity on \mathcal{M} . Then there exists time ψ on the oriented set \mathcal{M} , such, that:*

$$\mathbf{Y} = Y_\psi,$$

where Y_ψ is set of simultaneous states, generated by the time ψ .

Proof. Let \mathcal{M} be an oriented set and $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be a simultaneity on \mathcal{M} .

a) First we prove the theorem in the case, where the simultaneity \mathbf{Y} “separates” sequential unequal elementary states, that is where the following condition holds:

(Rp) For any $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $y \leftarrow x$ and $x \neq y$ there exists sets $A, B \in \mathbf{Y}$ such, that $x \in A$, $y \in B$ and $A \neq B$.

By lemma 5.1, (\mathbf{Y}, \leftarrow) is an oriented set. According to theorem 4.2, oriented set (\mathbf{Y}, \leftarrow) can be quasi one-point chronologized. Let $\Psi : \mathbf{T} \mapsto 2^{\mathbf{Y}}$ be quasi one-point time on (\mathbf{Y}, \leftarrow) . By definition 4.1 of quasi one-point time, for any $t \in \mathbf{T}$ the set $\Psi(t)$ is a singleton. This means, that:

$$\forall t \in \mathbf{T} \exists A_t \in \mathbf{Y} \quad \Psi(t) = \{A_t\}.$$

Denote:

$$\psi(t) := A_t, \quad t \in \mathbf{T}.$$

The next aim is to prove, that ψ is time on \mathcal{M} . Since ψ is time on \mathbf{Y} , then $\bigcup_{t \in \mathbf{T}} \Psi(t) = \mathbf{Y}$. And, taking into account, that $\Psi(t) = \{A_t\}$, $t \in \mathbf{T}$, we obtain $\{A_t \mid t \in \mathbf{T}\} = \mathbf{Y}$. Therefore, since the family of sets \mathbf{Y} is simultaneity on \mathcal{M} , we have, $\bigcup_{t \in \mathbf{T}} \psi(t) = \bigcup_{t \in \mathbf{T}} A_t = \bigcup_{A \in \mathbf{Y}} A = \mathfrak{Bs}(\mathcal{M})$. Hence, for any $x \in \mathfrak{Bs}(\mathcal{M})$ there exists a time moment $t \in \mathbf{T}$ such, that $x \in \psi(t)$. Thus, the first condition of the time definition 3.1 is satisfied. Now, we are going to prove, that the second condition of the definition 3.1 also is satisfied. Let $x, y \in \mathcal{M}$, $y \leftarrow x$ and $x \neq y$. By condition (Rp), there exist sets $A, B \in \mathbf{Y}$, such, that $x \in A$, $y \in B$ and $A \neq B$. Taking into account, that $x \in A$, $y \in B$ and $y \leftarrow x$, we obtain $B \leftarrow A$. Since $B \leftarrow A$, $A \neq B$ and Ψ — time on (\mathbf{Y}, \leftarrow) , there exist time moments $t, \tau \in \mathbf{T}$ such, that $A \in \Psi(t)$, $B \in \Psi(\tau)$ and $t < \tau$. And, taking into account $\Psi(t) = \{A_t\}$, $\Psi(\tau) = \{A_\tau\}$, we obtain $A = A_t$, $B = A_\tau$, that is $A = \psi(t)$, $B = \psi(\tau)$. Since $x \in A$, $y \in B$, then $x \in \psi(t)$, $y \in \psi(\tau)$, where $t < \tau$.

Thus, ψ is a time on \mathcal{M} . Moreover, taking into account what has been proven before, we get:

$$Y_\psi = \{\psi(t) \mid t \in \mathbf{T}\} = \{A_t \mid t \in \mathbf{T}\} = \mathbf{Y}.$$

Hence, in the case, when (Rp) is true, the theorem is proved.

b) Now we consider the case, when the condition (Rp) is not satisfied. Chose any element \tilde{x} , such, that $x \notin \mathfrak{Bs}(\mathcal{M})$. Denote:

$$\tilde{M} := \mathfrak{Bs}(\mathcal{M}) \cup \{\tilde{x}\}.$$

For elements $x, y \in \tilde{M}$ we put $y \widetilde{\leftarrow} x$ if and only if one of the following conditions is satisfied:

$$(a) \ x, y \in \mathfrak{Bs}(\mathcal{M}) \text{ and } y \leftarrow x; \quad (b) \ x = y = \tilde{x}.$$

That is the relation $\widetilde{\leftarrow}$ can be represented by formula $\widetilde{\leftarrow} = \leftarrow \cup \{(\tilde{x}, \tilde{x})\}$. Taking into account, that for $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ the condition $y \widetilde{\leftarrow} x$ is equivalent to the condition $y \leftarrow x$, further for relations $\widetilde{\leftarrow}$ and \leftarrow we will use the same denotation \leftarrow , assuming, that the relation \leftarrow is expanded to the set \tilde{M} . It is obvious, that (\tilde{M}, \leftarrow) is an oriented set. Denote:

$$\mathbf{Y}_0 := \{B \in \mathbf{Y} \mid \exists x, y \in B : x \neq y, y \leftarrow x\}.$$

Since the condition (Rp) is not satisfied, then $\mathbf{Y}_0 \neq \emptyset$. For $B \in \mathbf{Y}_0$ we put:

$$\tilde{B} := B \cup \{\tilde{x}\}.$$

Also we put:

$$\begin{aligned} \tilde{\mathbf{Y}}_0 &:= \{\tilde{B} \mid B \in \mathbf{Y}_0\} \\ \tilde{\mathbf{Y}} &:= \mathbf{Y} \cup \tilde{\mathbf{Y}}_0. \end{aligned}$$

Since \mathbf{Y} is a simultaneity on \mathcal{M} , and $\tilde{x} \in \tilde{B}$ for any set $\tilde{B} \in \tilde{\mathbf{Y}}_0$, then $\tilde{\mathbf{Y}}$ is a simultaneity on (\tilde{M}, \leftarrow) . The simultaneity $\tilde{\mathbf{Y}}$ readily satisfies the condition (Rp). Therefore, according to proven in paragraph a), there exists the time $\psi_1 : \mathbf{T} \mapsto 2^{\tilde{M}}$ on (\tilde{M}, \leftarrow) , such, that $Y_{\psi_1} = \{\psi_1(t) \mid t \in \mathbf{T}\} = \tilde{\mathbf{Y}}$. Now, we denote:

$$\psi(t) := \psi_1(t) \cap \mathfrak{B}\mathfrak{s}(\mathcal{M}), \quad t \in \mathbf{T}.$$

In accordance with the assertion 3.1, ψ is a time on \mathcal{M} . Moreover we obtain:

$$\begin{aligned} Y_\psi &= \{\psi(t) \mid t \in \mathbf{T}\} = \{\psi_1(t) \cap \mathfrak{B}\mathfrak{s}(\mathcal{M}) \mid t \in \mathbf{T}\} = \{A \cap \mathfrak{B}\mathfrak{s}(\mathcal{M}) \mid A \in \tilde{\mathbf{Y}}\} = \\ &= \{A \cap \mathfrak{B}\mathfrak{s}(\mathcal{M}) \mid A \in \mathbf{Y}\} \cup \{A \cap \mathfrak{B}\mathfrak{s}(\mathcal{M}) \mid A \in \tilde{\mathbf{Y}}_0\} = \\ &= \{A \mid A \in \mathbf{Y}\} \cup \{\tilde{B} \cap \mathfrak{B}\mathfrak{s}(\mathcal{M}) \mid B \in \mathbf{Y}_0\} = \mathbf{Y} \cup \{B \mid B \in \mathbf{Y}_0\} = \\ &= \mathbf{Y} \cup \mathbf{Y}_0 = \mathbf{Y}. \end{aligned}$$

□

Definition 5.3. Let $\mathbf{Y} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ be any simultaneity on the oriented set \mathcal{M} . Time ψ on \mathcal{M} will be named the **generating time** of the simultaneity \mathbf{Y} if and only if $\mathbf{Y} = Y_\psi$, where Y_ψ is the set of simultaneous states, generated by the time ψ .

Thus, the theorem 5.1 asserts, that any simultaneity always has a generating time. Below we consider the question about uniqueness of a generating time for a simultaneity (under the certain conditions). To ensure the correctness of staging this question, first of all, we need to introduce the concept of equivalence of two chronologizations.

Definition 5.4. Let \mathcal{M} be an oriented set and $\psi_1 : \mathbf{T}_1 \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$, $\psi_2 : \mathbf{T}_2 \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ be times for \mathcal{M} , defined on the linear ordered sets (\mathbf{T}_1, \leq_1) , (\mathbf{T}_2, \leq_2) . We say, that the chronologizations $\mathcal{H}_1 = ((\mathbf{T}_1, \leq_1), \psi_1)$ and $\mathcal{H}_2 = ((\mathbf{T}_2, \leq_2), \psi_2)$ are **equivalent** (using the denotation $\mathcal{H}_1 \uparrow\uparrow \mathcal{H}_2$) if and only if there exist an one-to-one correspondence $\xi : \mathbf{T}_1 \mapsto \mathbf{T}_2$ such, that:

- 1) ξ is order isomorphism between the linearly ordered sets (\mathbf{T}_1, \leq_1) , (\mathbf{T}_2, \leq_2) , that is for any $t, \tau \in \mathbf{T}_1$ the inequality $t \leq_1 \tau$ is equivalent to the inequality $\xi(t) \leq_2 \xi(\tau)$.
- 2) For any $t \in \mathbf{T}_1$ it is valid the equality $\psi_1(t) = \psi_2(\xi(t))$.

Assertion 5.2. Let \mathcal{M} be any oriented set and \mathcal{V} is any set, which consists of chronologizations of \mathcal{M} . Then the relation $\uparrow\uparrow$ is an equivalence relation on \mathcal{V} .

Proof. Throughout in this proof $\mathcal{H}_i = ((\mathbf{T}_i, \leq_i), \psi_i) \in \mathcal{V}$ ($i = 1, 2, 3$) mean any three chronologizations of the oriented set \mathcal{M} .

1) Reflexivity. Denote $\xi_{11}(t) := t$, $t \in \mathbf{T}_1$. It is obvious that ξ_{11} is a order isomorphism between (\mathbf{T}_1, \leq_1) and (\mathbf{T}_1, \leq_1) . Besides we have $\psi(t) = \psi(\xi(t))$, $t \in \mathbf{T}$. Thus $\mathcal{H}_1 \uparrow\uparrow \mathcal{H}_1$.

2) Symmetry. Let $\mathcal{H}_1 \uparrow\uparrow \mathcal{H}_2$. Then, by definition 5.4, there exist an one-to-one correspondence $\xi_{12} : \mathbf{T}_1 \mapsto \mathbf{T}_2$ such, that:

- 1) ξ_{12} is order isomorphism between the linearly ordered sets (\mathbf{T}_1, \leq_1) , (\mathbf{T}_2, \leq_2) .
- 2) $\psi_1(t) = \psi_2(\xi_{12}(t))$, for any $t \in \mathbf{T}_1$.

Since the mapping ξ_{12} is bijection, there exists the inverse mapping $\xi_{21}(t) := \xi_{12}^{[-1]}(t) \in \mathbf{T}_1$, $t \in \mathbf{T}_2$. Since ξ_{12} is order isomorphism between the linearly ordered sets (\mathbf{T}_1, \leq_1) , (\mathbf{T}_2, \leq_2) , then ξ_{21} is order isomorphism between (\mathbf{T}_2, \leq_2) and (\mathbf{T}_1, \leq_1) . Moreover, for any $t \in \mathbf{T}_2$ we obtain:

$$\psi_2(t) = \psi_2 \left(\xi_{12} \left(\xi_{12}^{[-1]}(t) \right) \right) = \psi_1 (\xi_{21}(t)).$$

Thus, $\mathcal{H}_2 \uparrow\uparrow \mathcal{H}_1$.

3) Transitivity. Let $\mathcal{H}_1 \uparrow\uparrow \mathcal{H}_2$, $\mathcal{H}_2 \uparrow\uparrow \mathcal{H}_3$. Then there exist order isomorphisms $\xi_{12} : \mathbf{T}_1 \mapsto \mathbf{T}_2$ and $\xi_{23} : \mathbf{T}_2 \mapsto \mathbf{T}_3$ such, that $\psi_1(t) = \psi_2(\xi_{12}(t))$, $t \in \mathbf{T}_1$ and $\psi_2(t) = \psi_3(\xi_{23}(t))$, $t \in \mathbf{T}_2$. Denote, $\xi_{13}(t) := \xi_{23}(\xi_{12}(t))$, $t \in \mathbf{T}_1$. It is easy to verify, that ξ_{13} is an order isomorphism between (\mathbf{T}_1, \leq_1) and (\mathbf{T}_3, \leq_3) . Moreover, for any $t \in \mathbf{T}_1$ we obtain:

$$\psi_1(t) = \psi_2(\xi_{12}(t)) = \psi_3(\xi_{23}(\xi_{12}(t))) = \psi_3(\xi_{13}(t)).$$

Therefore, $\mathcal{H}_1 \uparrow\uparrow \mathcal{H}_3$. □

Now, if we consider the question about uniqueness of a generating time for a simultaneity up to equivalence of corresponding chronologizations, the answer is still negative. For example we can consider a linearly ordered sets (\mathbf{T}, \leq) and (\mathbf{T}_1, \leq) such, that $\emptyset \neq \mathbf{T}_1 \subset \mathbf{T}$ (more accurately linear order relation on \mathbf{T}_1 is a restriction of order relation on \mathbf{T} , and both relations are denoted by the same symbol “ \leq ”). If $\psi_1 : \mathbf{T}_1 \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ is a time on the oriented set \mathcal{M} , then for any element $t_1 \in \mathbf{T}_1$ we can define the time:

$$\psi(t) := \begin{cases} \psi_1(t), & t \in \mathbf{T}_1; \\ \psi_1(t_1), & t \in \mathbf{T} \setminus \mathbf{T}_1, \end{cases}$$

This time is such, that $Y_\psi = Y_{\psi_1}$, although, in the case, when the ordered sets (\mathbf{T}, \leq) and (\mathbf{T}_1, \leq) are not isomorphic, the chronologizations $((\mathbf{T}, \leq), \psi)$ and $((\mathbf{T}_1, \leq), \psi_1)$ are not equivalent. That is why, to obtain the positive answer for the above question, further we will impose additional conditions on simultaneity and generating time.

Definition 5.5. Let \mathcal{M} be an oriented set.

1) We will say, that a set $B \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{M})$ is **monotonously sequential** to the set $A \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ in the oriented set \mathcal{M} if and only if there exist elements $x \in A$ and $y \in B$ such, that $y \xleftarrow{\mathcal{M}} x$ and $x \not\xleftarrow{\mathcal{M}} y$. In this case we will use the denotation $B \xleftarrow{(\mathfrak{m})}_{\mathcal{M}} A$.

2) Let $\mathcal{Q} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ be any system of subsets of $\mathfrak{B}\mathfrak{s}(\mathcal{M})$. We will say, that a set $A \in \mathcal{Q}$ is **transitively monotonously sequential** to a set $B \in \mathcal{Q}$ relative to the system \mathcal{Q} if and only if there exist a finite sequence of sets $C_0, C_1, \dots, C_n \in \mathcal{Q}$ ($n \in \mathbb{N}$) such, that $C_0 = A$, $C_n = B$ and $C_k \xleftarrow{(\mathfrak{m})}_{\mathcal{M}} C_{k-1}$, for any $k \in \overline{1, n}$ (where $\overline{1, n} = \{1, \dots, n\}$). In this case we will use the denotation $B \xleftarrow{(\mathfrak{m})}_{\mathcal{M}}^{\mathcal{Q}} A$.

In the case where the oriented set \mathcal{M} is known in advance, the char \mathcal{M} in the denotations $\leftarrow_{\mathcal{M}}^{(m)}$ and $\leftarrow_{\mathcal{M}}^{\mathcal{Q}}^{(m)}$ will be released, and we will use denotations $\leftarrow^{(m)}$ and $\leftarrow^{\mathcal{Q}}^{(m)}$ instead.

Remark 5.1. It is easy to prove, that for any system of sets $\mathcal{Q} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ (in any oriented set \mathcal{M}) the binary relation $\leftarrow^{\mathcal{Q}}^{(m)}$ is transitive on the set \mathcal{Q} .

Assertion 5.3. *Let \mathcal{M} be an oriented set, and $\mathfrak{S}, \mathfrak{S}' \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ are systems of subsets of $\mathfrak{B}\mathfrak{s}(\mathcal{M})$, moreover $\mathfrak{S} \subseteq \mathfrak{S}'$ (this means, that for any set $A \in \mathfrak{S}$ there exist a set $A' \in \mathfrak{S}'$ such, that $A \subseteq A'$).*

Then for any $A, B \in \mathfrak{S}$ and $A', B' \in \mathfrak{S}'$ such, that $A \subseteq A'$, $B \subseteq B'$ correlation $B \leftarrow_{(m)}^{\mathfrak{S}} A$ involves the correlation $B' \leftarrow_{(m)}^{\mathfrak{S}'} A'$.

Proof. Suppose that the conditions of the assertion are performed. Let $A, B \in \mathfrak{S}$, $A', B' \in \mathfrak{S}'$, $A \subseteq A'$, $B \subseteq B'$ and $B \leftarrow_{(m)}^{\mathfrak{S}} A$. Then, there exists a finite sequence of sets $C_0, \dots, C_n \in \mathfrak{S}$ ($n \in \mathbb{N}$) such, that $C_0 = A$, $C_n = B$ and $C_k \leftarrow_{(m)} C_{k-1}$ (for any $k \in \overline{1, n}$). Since $\mathfrak{S} \subseteq \mathfrak{S}'$, there exist sets $C'_0, \dots, C'_n \in \mathfrak{S}'$ such, that $C_k \subseteq C'_k$ ($k \in \overline{0, n}$). Moreover, since $C_0 = A \subseteq A' \in \mathfrak{S}'$, $C_n = B \subseteq B' \in \mathfrak{S}'$, we can believe that $C'_0 = A'$, $C'_n = B'$. Taking into account that $C_k \subseteq C'_k$ ($k \in \overline{0, n}$), and $C_k \leftarrow_{(m)} C_{k-1}$ ($k \in \overline{1, n}$), we obtain $C'_k \leftarrow_{(m)} C'_{k-1}$, $k \in \overline{1, n}$ (where $C'_0 = A'$, $C'_n = B'$). Thus $B' \leftarrow_{(m)}^{\mathfrak{S}'} A'$. \square

Definition 5.6. *Let \mathcal{M} be an oriented set.*

1) *System of sets $\mathfrak{S} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ will be referred to as **unrepeatable** if and only if there not exist sets $A, B \in \mathfrak{S}$ such, that $A \leftarrow_{(m)}^{\mathfrak{S}} B$ and $B \leftarrow_{(m)}^{\mathfrak{S}} A$. In particular, in the case, where a simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ is unrepeatable system of sets, this simultaneity we will call an **unrepeatable simultaneity**.*

2) *Simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ will be referred to as **precise** if and only if for any $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ such, that $y \leftarrow x$ and $x \neq y$ there exist sets $A, B \in \mathbf{Y}$ such, that $x \in A$, $y \in B$, $A \neq B$ and $B \leftarrow_{(m)}^{\mathbf{Y}} A$ (this means, that this simultaneity “fixes” all changes on the oriented set \mathcal{M}).*

3) *Simultaneity \mathbf{Y} will be called **precisely-unrepeatable** if and only if it is precise and, at the same time, unrepeatable.*

Remark 5.2. From the remark 5.1 it readily follows, that in the case, where a simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ is unrepeatable, the relation $\leftarrow_{(m)}^{\mathbf{Y}}$ is a strict order on \mathbf{Y} (ie $\leftarrow_{(m)}^{\mathbf{Y}}$ is anti-reflexive and transitive relation).

Assertion 5.4. *Let \mathcal{M} be an oriented set and $\mathfrak{S} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ is unrepeatable system of sets. Then:*

- 1) *For any $A, B \in \mathfrak{S}$, such, that $B \leftarrow_{(m)}^{\mathfrak{S}} A$ is true $A \neq B$.*
- 2) *If $\mathfrak{S}_1 \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ and $\mathfrak{S}_1 \subseteq \mathfrak{S}$, then \mathfrak{S}_1 also is unrepeatable system of sets.*

Proof. 1) Let $\mathfrak{S} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ be unrepeatable system of sets. If we suppose, that $B \leftarrow_{(m)}^{\mathfrak{S}} A$ and $A \leftarrow_{(m)}^{\mathfrak{S}} B$ (for some $A, B \in \mathfrak{S}$), then we obtain $A \leftarrow_{(m)}^{\mathfrak{S}} B$ and $B \leftarrow_{(m)}^{\mathfrak{S}} A$, which is impossible, since the system of sets \mathfrak{S} is unrepeatable.

2) Let $\mathfrak{S}_1 \subseteq \mathfrak{S}$. Suppose, that the system of sets \mathfrak{S}_1 is not unrepeatable. Then, there exists sets $A_1, B_1 \in \mathfrak{S}_1$ such, that $A_1 \leftarrow_{(m)}^{\mathfrak{S}_1} B_1$ and $B_1 \leftarrow_{(m)}^{\mathfrak{S}_1} A_1$. Since $\mathfrak{S}_1 \subseteq \mathfrak{S}$, there exist sets $A, B \in \mathfrak{S}$ such, that $A_1 \subseteq A$, $B_1 \subseteq B$. Hence, by assertion 5.3, we obtain $A \leftarrow_{(m)}^{\mathfrak{S}} B$ and $B \leftarrow_{(m)}^{\mathfrak{S}} A$, which is impossible, since the system of sets \mathfrak{S} is unrepeatable. Thus, the system of sets \mathfrak{S}_1 is unrepeatable, because the opposite assumption is wrong. \square

Lemma 5.2. Let $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}(\mathcal{M})}$ be a monotone time on an oriented set \mathcal{M} , and Y_ψ be a simultaneity, generated by the time ψ . Then for any $t_1, t_2 \in \mathbf{T}$ the condition $\psi(t_2) \xleftarrow{Y_\psi(m)} \psi(t_1)$ leads to $t_1 < t_2$.

Proof. 1) First we consider the case, where $\psi(t_2) \xleftarrow{(m)} \psi(t_1)$. In this case, by definition 5.5, there exist elements $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ such, that $x_2 \leftarrow x_1$ and $x_1 \not\leftarrow x_2$. Hence, since the time ψ is monotone, we obtain $t_1 < t_2$ (by the definition 4.1).

Now, we consider the general case, $\psi(t_2) \xleftarrow{Y_\psi(m)} \psi(t_1)$. In this case, by definition 5.5, there exist time points $\tau_0, \tau_1, \dots, \tau_n \in \mathbf{T}$ such, that $\tau_0 = t_1$, $\tau_n = t_2$ and $\psi(\tau_k) \xleftarrow{(m)} \psi(\tau_{k-1})$ for any $k \in \overline{1, n}$. By statement 1), $\tau_{k-1} < \tau_k$, $k \in \overline{1, n}$. Thus, $t_1 = \tau_0 < \tau_1 < \dots < \tau_n = t_2$. \square

Definition 5.7. We will say, that a simultaneity \mathbf{Y} on an oriented set is **monotone-connected** if and only if for any sets $A, B \in \mathbf{Y}$ such, that $A \neq B$ it holds one of the conditions $A \xleftarrow{\mathbf{Y}(m)} B$ or $B \xleftarrow{\mathbf{Y}(m)} A$.

Remark 5.3. Directly from the definition 5.7 and remark 5.2 it follows, that if a simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{B}(\mathcal{M})}$ is unrepeatable and monotone-connected, then the relation $\xleftarrow{\mathbf{Y}(m)}$ is a strict linear order on \mathbf{Y} .

Definition 5.8. Let \mathcal{M} be an oriented set and (\mathbf{T}, \leq) be a linear ordered set. Time $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}(\mathcal{M})}$ will be called **incessant** if and only if there not exist time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$ and for any $t \in \mathbf{T}$, $t_1 \leq t \leq t_2$ it is true the equality $\psi(t) = \psi(t_1)$. In the case, where the time ψ is both monotone and incessant it will be called **strictly monotone**.

Lemma 5.3. Let \mathbf{Y} be precisely-unrepeatable and monotone-connected simultaneity on the oriented set \mathcal{M} and $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}(\mathcal{M})}$ is the time, generating this simultaneity.

1) If the time ψ is strictly monotone, then it is **unrepeatable** (this means, that for any $t_1, t_2 \in \mathbf{T}$ such, that $t_1 \neq t_2$ the correlation $\psi(t_1) \neq \psi(t_2)$ is valid).

2) The time ψ is strictly monotone if and only if for any $t_1, t_2 \in \mathbf{T}$ inequality $t_1 < t_2$ implies the correlation $\psi(t_2) \xleftarrow{\mathbf{Y}(m)} \psi(t_1)$.

3) If the time ψ is strictly monotone, then the strictly linearly ordered sets $(\mathbf{T}, >)$ and $(\mathbf{Y}, \xleftarrow{\mathbf{Y}(m)})$ are isomorphic relative the order, and the mapping $\psi : \mathbf{T} \mapsto \mathbf{Y}$ is the order isomorphism between them.

Proof. 1) Let, under conditions of the lemma, time $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}(\mathcal{M})}$ be strictly monotone. Suppose, there exist time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$ and $\psi(t_1) = \psi(t_2)$. Since the time ψ (being strictly monotone) is incessant, there exists a time point $t_3 \in \mathbf{T}$ such, that $t_1 < t_3 < t_2$ and $\psi(t_3) \neq \psi(t_1) = \psi(t_2)$. So far as $\psi(t_3) \neq \psi(t_1)$ and the simultaneity \mathbf{Y} is monotone-connected, one of the conditions $\psi(t_3) \xleftarrow{\mathbf{Y}(m)} \psi(t_1)$ or $\psi(t_1) \xleftarrow{\mathbf{Y}(m)} \psi(t_3)$ is performed. But since $t_1 < t_3$ the correlation $\psi(t_1) \xleftarrow{\mathbf{Y}(m)} \psi(t_3)$ is impossible by lemma 5.2. Therefore, $\psi(t_3) \xleftarrow{\mathbf{Y}(m)} \psi(t_1)$. Similarly, since $t_3 < t_2$ and $\psi(t_3) \neq \psi(t_2)$, we obtain $\psi(t_2) \xleftarrow{\mathbf{Y}(m)} \psi(t_3)$. Hence, taking into account, that $\psi(t_1) = \psi(t_2)$, we have $\psi(t_3) \xleftarrow{\mathbf{Y}(m)} \psi(t_1)$ and $\psi(t_1) \xleftarrow{\mathbf{Y}(m)} \psi(t_3)$, which is impossible, because the simultaneity $\mathbf{Y} = Y_\psi$ is unrepeatable.

2.a) Suppose, that the time $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}(\mathcal{M})}$ is strictly monotone. Chose any time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$. By the first statement of this lemma, $\psi(t_1) \neq \psi(t_2)$. Since the simultaneity \mathbf{Y} is monotone-connected, one of the conditions $\psi(t_2) \xleftarrow{\mathbf{Y}(m)} \psi(t_1)$ or $\psi(t_1) \xleftarrow{\mathbf{Y}(m)} \psi(t_2)$ is performed. But, so far as $t_1 < t_2$, the condition $\psi(t_1) \xleftarrow{\mathbf{Y}(m)} \psi(t_2)$ is impossible by lemma 5.2. Thus:

$$\forall t_1, t_2 \in \mathbf{T} \ t_1 < t_2 \Rightarrow \psi(t_2) \xleftarrow[\text{(m)}]{\mathbf{Y}} \psi(t_1). \quad (5)$$

Now we suppose, that the condition (5) holds. The first aim is to prove, that the time ψ is monotone. Consider any elementary states $x_1, x_2 \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ such, that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$, $x_2 \leftarrow x_1$ and $x_1 \not\leftarrow x_2$ (where $t_1, t_2 \in \mathbf{T}$). By definition 5.5, $\psi(t_2) \xleftarrow[\text{(m)}]{\mathbf{Y}} \psi(t_1)$. Consequently,

$$\psi(t_2) \xleftarrow[\text{(m)}]{\mathbf{Y}} \psi(t_1). \quad (6)$$

If we suppose $t_1 \geq t_2$, we must obtain:

$$\psi(t_1) \xleftarrow[\text{(m)}]{\mathbf{Y}} \psi(t_2) \quad (7)$$

(indeed, in the case $t_1 = t_2$ the correlation (7) follows from (6), and in the case $t_1 > t_2$ the correlation (7) is caused by the condition (5)). Thus, in the case $t_1 \geq t_2$, both of the conditions (6) and (7) must be performed, which is impossible (since the simultaneity \mathbf{Y} is unrepeatable). Consequently, only the inequality $t_1 < t_2$ is possible. This proves that the time ψ is monotone.

Thus, it remains to prove, that the time ψ is incessant. Suppose, there exist time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$, and $\psi(t) = \psi(t_1)$ for any $t \in \mathbf{T}$, satisfying $t_1 \leq t \leq t_2$. Then, in particular, $\psi(t_1) = \psi(t_2)$ (where $t_1 < t_2$). Since $t_1 < t_2$, by condition (5), the correlation (6) must be performed. But since $\psi(t_1) = \psi(t_2)$, the correlation (7) also is performed, which is impossible (since the simultaneity \mathbf{Y} is unrepeatable). Therefore, the time ψ is incessant.

Thus, the time ψ is strictly monotone.

3) Let the time $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ be strictly monotone. According to the first statement of the lemma, the mapping $\psi : \mathbf{T} \mapsto \mathbf{Y} = Y_\psi$ is one-to-one correspondence between \mathbf{T} and $\mathbf{Y} = Y_\psi$. According to the second statement of the lemma, for any $t_1, t_2 \in \mathbf{T}$ the inequality $t_2 > t_1$ implies the correlation $\psi(t_2) \xleftarrow[\text{(m)}]{\mathbf{Y}} \psi(t_1)$. Hence, taking into account, that by remark 5.3, $\left(\mathbf{Y}, \xleftarrow[\text{(m)}]{\mathbf{Y}}\right)$ is a linear ordered set (with strict order), we conclude, that the mapping ψ is isomorphism between the strictly linear ordered sets $(\mathbf{T}, >)$ and $\left(\mathbf{Y}, \xleftarrow[\text{(m)}]{\mathbf{Y}}\right)$. \square

Remark 5.4. It turns out, that for any precisely-unrepeatable and monotone-connected simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ the assertion, inverse to the first statement, of the lemma 5.3, in general, is not true. To demonstrate this we present the following example.

Example 5.1. Let us consider the following oriented set:

$$\begin{aligned} \mathfrak{B}\mathfrak{s}(\mathcal{M}) &:= \{1, 2, 3, 4\}; \\ \xleftarrow[\mathcal{M}]{} &:= \{(3, 1), (4, 2)\} \cup \text{diag}(\mathfrak{B}\mathfrak{s}(\mathcal{M})), \end{aligned}$$

that is, in the other words, $3 \leftarrow 1$, $4 \leftarrow 2$, $1 \leftarrow 1$, $2 \leftarrow 2$, $3 \leftarrow 3$, $4 \leftarrow 4$. In this oriented set we consider the following simultaneity:

$$\mathbf{Y} := \{\{1, 2\}, \{3, 4\}, \{2, 3\}\}.$$

Then, we have $\{2, 3\} \xleftarrow[\text{(m)}]{} \{1, 2\}$, $\{3, 4\} \xleftarrow[\text{(m)}]{} \{2, 3\}$, $\{3, 4\} \xleftarrow[\text{(m)}]{} \{1, 2\}$, and $\{2, 3\} \not\xleftarrow[\text{(m)}]{} \{3, 4\}$, $\{1, 2\} \not\xleftarrow[\text{(m)}]{} \{2, 3\}$, $\{1, 2\} \not\xleftarrow[\text{(m)}]{} \{3, 4\}$, moreover, any set of simultaneity \mathbf{Y} is not monotonously sequential by the itself. That is, schematically:

$$\begin{array}{ccccc} \{3, 4\} & \xleftarrow[\text{(m)}]{} & \{2, 3\} & \xleftarrow[\text{(m)}]{} & \{1, 2\} \\ \nwarrow & < \text{---} & \xleftarrow[\text{(m)}]{} & < \text{---} & \swarrow \end{array},$$

and, moreover, the relation “ $\leftarrow_{(m)}$ ” on the simultaneity \mathbf{Y} is fully generated by the last scheme. And from this scheme it is evident, that the simultaneity \mathbf{Y} is unrepeatable and monotone-connected. Moreover, it is easy to verify, that this simultaneity is precise.

Also we consider the following linear ordered set:

$$\mathbf{T} := \{1, 2, 3\},$$

with the standard linear order relation on the natural numbers (\leq). The simultaneity \mathbf{Y} can be generated by the following times:

$$\begin{aligned} \psi_1 : \quad \psi_1(1) &:= \{1, 2\}, \quad \psi_1(2) := \{2, 3\}, \quad \psi_1(3) := \{3, 4\}; \\ \psi_2 : \quad \psi_2(1) &:= \{1, 2\}, \quad \psi_2(2) := \{3, 4\}, \quad \psi_2(3) := \{2, 3\}. \end{aligned}$$

Both of times ψ_1 and ψ_2 are, evidently, unrepeatable, but the time ψ_2 is not monotone, because of:

$$\begin{aligned} 2 &\in \psi_2(3), \quad 4 \in \psi_2(2), \\ 4 &\leftarrow 2, \quad 2 \not\leftarrow 4, \quad \text{but } 3 \not\leftarrow 2. \end{aligned}$$

Theorem 5.2. *For any precisely-unrepeatable and monotone-connected simultaneity \mathbf{Y} an unique up to equivalence of chronologizations strictly monotone time ψ exists, such, that $\mathbf{Y} = Y_\psi$.*

It should be noted, that the uniqueness up to equivalence of chronologizations in the theorem 5.2 is understood as follows:

“if strictly monotone times ψ_1 and ψ_2 , defined on linear ordered sets \mathbb{T}_1 and \mathbb{T}_2 are such, that $\mathbf{Y} = Y_{\psi_1} = Y_{\psi_2}$, then $\mathcal{H}_1 \uparrow\uparrow \mathcal{H}_2$, where \mathcal{H}_1 and \mathcal{H}_2 are corresponding chronologizations ($\mathcal{H}_i = (\mathbb{T}_i, \psi_i)$, $i \in \{1, 2\}$)”.

Proof. 1. Let \mathbf{Y} be precisely-unrepeatable and monotone-connected simultaneity on an oriented set \mathcal{M} . Then, by remark 5.3, $\leftarrow_{(m)}^{\mathbf{Y}}$ is a strict linear order on \mathbf{Y} . Hence, the relation $\rightarrow_{(m)}^{\mathbf{Y}}$, being inverse to the relation $\leftarrow_{(m)}^{\mathbf{Y}}$, also is a strict linear order on \mathbf{Y} . Denote:

$$\mathbf{T} := \mathbf{Y}.$$

For $t, \tau \in \mathbf{T} = \mathbf{Y}$ we will assume, that $t \leq \tau$ if and only if:

$$t = \tau \text{ or } t \rightarrow_{(m)}^{\mathbf{Y}} \tau.$$

That is \leq is (non-strict) linear order, generated by the strict order $\rightarrow_{(m)}^{\mathbf{Y}}$. Therefore, for $t, \tau \in \mathbf{T}$ the following logical equivalence is true:

$$t < \tau \quad \text{if and only if} \quad t \rightarrow_{(m)}^{\mathbf{Y}} \tau, \tag{8}$$

where record means, that $t \leq \tau$ and $t \neq \tau$. Thus, (\mathbf{T}, \leq) is a linear ordered set. Denote:

$$\psi(t) := t, \quad t \in \mathbf{T} = \mathbf{Y}.$$

Since $\mathbf{T} = \mathbf{Y}$, then $\psi(t) = t \in \mathbf{Y} \subseteq 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ for $t \in \mathbf{T}$.

2. The next aim is to prove, that ψ is a time on \mathcal{M} .

(a) Since \mathbf{Y} is a simultaneity, then for any $x \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ there exists set $t_x \in \mathbf{Y} = \mathbf{T}$, such, that $x \in t_x$. Therefore, we obtain $\psi(t_x) = t_x \ni x$. Thus, the first condition of the time definition 3.1 is performed.

(b) Suppose, that $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$, $y \leftarrow x$ and $x \neq y$. Since the simultaneity \mathbf{Y} is precise, there exist $t_x, t_y \in \mathbf{Y} = \mathbf{T}$ such, that $x \in t_x$, $y \in t_y$ and $t_y \stackrel{\mathbf{Y}}{\leftarrow(m)} t_x$. Then, by (8), $t_x < t_y$. Moreover, since $\psi(t) = t$, $t \in \mathbf{T}$, we have:

$$x \in t_x = \psi(t_x); \quad y \in t_y = \psi(t_y).$$

Consequently, the second condition of the definition 3.1 also is satisfied.

Thus, the mapping ψ is a time.

3. Now we aim to prove, that the time ψ is strictly monotone.

(a) Let $x \in \psi(t_x)$, $y \in \psi(t_y)$, $y \leftarrow x$ and $x \not\leftarrow y$. Then $t_y = \psi(t_y) \leftarrow(m) \psi(t_x) = t_x$. Consequently, $t_y \stackrel{\mathbf{Y}}{\leftarrow(m)} t_x$, ie, by (8), $t_x < t_y$. Thus, the time ψ is monotone.

(b) Suppose, that this time is not incessant. Then there exist $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$ and $\psi(t) = \psi(t_1)$ for any $t \in \mathbf{T}$, satisfying the inequality $t_1 \leq t \leq t_2$. In particular this means, that $\psi(t_2) = \psi(t_1)$. And, since $\psi(\tau) = \tau$, $\tau \in \mathbf{T}$, we obtain $t_2 = t_1$, which contradicts the inequality $t_1 < t_2$. Therefore, the assumption is wrong, and the time ψ is incessant.

Thus, the time ψ is strictly monotone.

4. It remains to prove, that the time ψ is unique up to equivalence of chronologizations. Let $\psi_1 : \mathbf{T}_1 \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ be an other strictly monotone time such, that $Y_{\psi_1} = \mathbf{Y}$ (where (\mathbf{T}_1, \leq_1) is a linear ordered set. Then, by lemma 5.3, the linear ordered (by strict order) sets $(\mathbf{T}_1, >_1)$ and $(\mathbf{Y}, \stackrel{\mathbf{Y}}{\leftarrow(m)})$ are isomorphic relative the order, with the mapping $\psi_1 : \mathbf{T}_1 \mapsto \mathbf{Y}$ being isomorphism, where $>_1$ is relation, inverse to the relation $<_1$, and $<_1$ is strict order, generated by non-strict order \leq_1 . Thus, the ordered sets (\mathbf{T}_1, \leq_1) and $(\mathbf{Y}, \leq) = (\mathbf{T}, \leq)$ also are isomorphic with the isomorphism ψ_1 . Moreover, for any $t \in \mathbf{T}_1$, we have:

$$\psi_1(t) = \psi(\psi_1(t)),$$

ie, by definition 5.4, $((\mathbf{T}_1, \leq_1), \psi_1) \uparrow\uparrow ((\mathbf{T}, \leq), \psi)$. □

Definition 5.9. Let \mathcal{M} be an oriented set, and $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ is a time on \mathcal{M} .

A mapping $\mathbf{h} : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ will be referred to as **chronometric process** (for the time ψ), if and only if:

- 1) $\mathbf{h}(t) \subseteq \psi(t)$ for any $t \in \mathbf{T}$.
- 2) For arbitrary $t, \tau \in \mathbf{T}$ inequality $t < \tau$ is valid if and only if $\mathbf{h}(\tau) \stackrel{\mathbf{h}(\mathbf{T})}{\leftarrow(m)} \mathbf{h}(t)$ and $\mathbf{h}(t) \neq \mathbf{h}(\tau)$, where $\mathbf{h}(\mathbf{T}) = \{\mathbf{h}(t) \mid t \in \mathbf{T}\}$;

The time ψ on \mathcal{M} will be referred to as **internal** if and only if there exists at least one chronometric process for this time.

Sense of the term “internal time” lies in the fact that in the case, where a time on a primitive changeable set is internal, this time can be measured within this primitive changeable set, using the chronometric process as a “clock” and states of this process as “indicators of time points”. The next aim is to establish the sufficient condition of existence and uniqueness of internal time for given simultaneity.

Lemma 5.4. The generating time for precisely-unrepeatable and monotone-connected simultaneity is internal if and only if it is strictly monotone.

Proof. Let \mathcal{M} be an oriented set, \mathbf{Y} is precisely-unrepeatable and monotone-connected simultaneity and $\psi : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ is a time on \mathcal{M} , which generates \mathbf{Y} (ie $\mathbf{Y} = Y_\psi$).

1) Suppose, that the time ψ is internal. Then there exists a chronometric process $\mathbf{h} : \mathbf{T} \mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{M})}$ for the time ψ .

1.a) First we are going to prove, that the time ψ is monotone. Let $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$, $x_2 \leftarrow x_1$ and $x_1 \not\leftarrow x_2$. Then $\psi(t_2) \leftarrow_{(m)} \psi(t_1)$, ie $\psi(t_2) \leftarrow_{(m)}^{\mathbf{Y}} \psi(t_1)$. Hence, since the simultaneity \mathbf{Y} is unrepeatable, using the assertion 5.4, we obtain $\psi(t_1) \neq \psi(t_2)$, ie $t_1 \neq t_2$. Thus, the possible cases are $t_1 < t_2$ or $t_2 < t_1$. Let us suppose, that $t_2 < t_1$. Then, since \mathbf{h} is chronometric process, we have, $\mathbf{h}(t_1) \leftarrow_{(m)}^{\mathbf{h}(\mathbf{T})} \mathbf{h}(t_2)$. From definition 5.9 it follows, that $\mathbf{h}(\mathbf{T}) \sqsubseteq \mathbf{Y}$, consequently, using the assertion 5.3, we obtain $\psi(t_1) \leftarrow_{(m)}^{\mathbf{Y}} \psi(t_2)$, which is impossible, because the simultaneity \mathbf{Y} is unrepeatable, and, according to the above proved, $\psi(t_2) \leftarrow_{(m)}^{\mathbf{Y}} \psi(t_1)$. So only possible it remains the inequality $t_1 < t_2$, which proves, that the time ψ is monotone.

1.b) Now, we are going to prove, that the time ψ is incessant. Assume the contrary. Then there exist the time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$, and for any $t \in \mathbf{T}$, satisfying $t_1 \leq t \leq t_2$, the equality $\psi(t) = \psi(t_1)$ is true. Then, in particular, $\psi(t_2) = \psi(t_1)$. But, since \mathbf{h} is chronometric process, then $\mathbf{h}(t_2) \leftarrow_{(m)}^{\mathbf{h}(\mathbf{T})} \mathbf{h}(t_1)$, and, by assertion 5.3, $\psi(t_2) \leftarrow_{(m)}^{\mathbf{Y}} \psi(t_1)$. Therefore, by assertion 5.4, $\psi(t_2) \neq \psi(t_1)$, which contradicts the above written. Thus, the time ψ is incessant. And, taking into account what has been proved in paragraph 1.a), we have, that time ψ is strictly monotone.

2) Now we suppose, that the time ψ is strictly monotone. Then, by lemma 5.3, the strictly linear ordered sets $(\mathbf{T}, >)$ and $\left(\mathbf{Y}, \leftarrow_{(m)}^{\mathbf{Y}}\right) = \left(\mathbf{Y}, \leftarrow_{(m)}^{Y_\psi}\right)$ are isomorphic relative the order, and the mapping $\psi : \mathbf{T} \mapsto \mathbf{Y}$ is the order isomorphism between them. That is why, for ant $t_1, t_2 \in \mathbf{T}$ the conditions $t_1 < t_2$ and $\psi(t_2) \leftarrow_{(m)}^{Y_\psi} \psi(t_1)$ are logically equivalent (where $Y_\psi = \mathbf{Y} = \psi(\mathbf{T})$). Thus, taking into account statement 1 of the assertion 5.4, we conclude, that the mapping $\mathbf{h}(t) = \psi(t)$, $t \in \mathbf{T}$ is a chronometric process for the time ψ . Consequently, the time ψ is internal. \square

The next theorem follows from the lemma 5.4 and theorem 5.2.

Theorem 5.3. *For any precisely-unrepeatable and monotone-connected simultaneity \mathbf{Y} an unique up to equivalence of chronologizations internal time ψ exists, such, that $\mathbf{Y} = Y_\psi$.*

Philosophical content of the theorem 5.3 is that the originality of pictures of reality, possibility to see any changes in the sequential simultaneous states, and connectivity of different pictures of reality by chains of changes are uniquely generating the course of “internal” time in “our” world.

Remark 5.5. Further we will denote primitive changeable sets by large calligraphic letters.

Let $\mathcal{P} = (\mathcal{M}, \mathbb{T}, \phi)$ be a primitive changeable set, where $\mathbb{T} = (\mathbf{T}, \leq)$ is a linear ordered set. We introduce the following denotations:

$$\begin{aligned} \mathfrak{B}\mathfrak{s}(\mathcal{P}) &:= \mathfrak{B}\mathfrak{s}(\mathcal{M}); & \leftarrow_{\mathcal{P}} &:= \leftarrow_{\mathcal{M}}; \\ \mathbf{T}\mathbf{m}(\mathcal{P}) &:= \mathbf{T}; & \leq_{\mathcal{P}} &:= \leq; & \psi_{\mathcal{P}} &:= \phi. \end{aligned}$$

Also we will use the records $\geq_{\mathcal{P}}, <_{\mathcal{P}}, >_{\mathcal{P}}$ to denote the inverse, strict and strict inverse orders, generated by the order $\leq_{\mathcal{P}}$. The set $\mathfrak{B}\mathfrak{s}(\mathcal{P})$ we will name a *basic* set or a set of all *elementary states* of the primitive changeable set \mathcal{P} and we will denote it by $\mathfrak{B}\mathfrak{s}(\mathcal{M})$. Elements of the set $\mathfrak{B}\mathfrak{s}(\mathcal{P})$ will be named the *elementary states* of \mathcal{P} . The relation $\leftarrow_{\mathcal{P}}$ we will name a *directing relation of changes* of \mathcal{P} . The set $\mathbf{T}\mathbf{m}(\mathcal{P})$ will be named the *set of time points* of \mathcal{P} . The relations $\leq_{\mathcal{P}}, <_{\mathcal{P}}, \geq_{\mathcal{P}}, >_{\mathcal{P}}$ will be referred to as the relations of non-strict, strict, non-strict inverse and strict inverse time order on \mathcal{P} .

In the case, where the primitive changeable set \mathcal{P} , in question is clear, in the notations $\overset{\mathcal{P}}{\leftarrow}$, $\leq_{\mathcal{P}}$, $<_{\mathcal{P}}$, $\geq_{\mathcal{P}}$, $>_{\mathcal{P}}$, $\psi_{\mathcal{P}}$ the symbol \mathcal{P} will be omitted, and the notations \leftarrow , \leq , $<$, \geq , $>$, ψ will be used instead.

6 Systems of Abstract Trajectories and Primitive Changeable Sets, Generated by them

Definition 6.1. Let M be an arbitrary set and $\mathbb{T} = (\mathbf{T}, \leq)$ is any linear ordered set.

1. Any mapping $r : \mathfrak{D}(r) \mapsto M$, where $\mathfrak{D}(r) \subseteq \mathbf{T}$, will be referred to as an **abstract trajectory** from \mathbb{T} to M (here $\mathfrak{D}(r)$ is the domain of the abstract trajectory r).
2. Any set \mathcal{R} , which consists of abstract trajectories from \mathbb{T} to M and satisfies:

$$\bigcup_{r \in \mathcal{R}} \mathfrak{R}(r) = M$$

will be named **system of abstract trajectories** from \mathbb{T} to M (here $\mathfrak{R}(r)$ is the range of the abstract trajectory r).

Theorem 6.1. Let \mathcal{R} be a system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to M . Then there exists a unique primitive changeable set \mathcal{P} , which satisfies the following conditions:

- 1) $\mathfrak{B}\mathfrak{s}(\mathcal{P}) = M$; $\mathbf{T}\mathfrak{m}(\mathcal{P}) = \mathbf{T}$, $\leq_{\mathcal{P}} = \leq$.
- 2) For any $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{P})$ the condition $y \leftarrow x$ is satisfied if and only if there exist an abstract trajectory $r = r_{x,y} \in \mathcal{R}$ and elements $t, \tau \in \mathfrak{D}(r) \subseteq \mathbf{T}$ such, that $x = r(t)$, $y = r(\tau)$ and $t \leq \tau$.
- 3) For arbitrary $x \in \mathfrak{B}\mathfrak{s}(\mathcal{P})$ and $t \in \mathbf{T}\mathfrak{m}(\mathcal{P})$ the condition $x \in \psi(t)$ is satisfied if and only if there exist an abstract trajectory $r = r_x \in \mathcal{R}$ such, that $t \in \mathfrak{D}(r)$ and $x = r(t)$.

Proof. Let \mathcal{R} be any system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to M . Define the following relation:

$$\overset{\mathcal{R}}{\leftarrow} = \{(y, x) \in M \times M \mid \exists r \in \mathcal{R} \exists t, \tau \in \mathfrak{D}(r) : x = r(t), y = r(\tau), t \leq \tau\}$$

on the set M (where the symbol \times denotes the Cartesian product of sets). Or, in other words, for $x, y \in M$ the correlation $y \overset{\mathcal{R}}{\leftarrow} x$ is true if and only if there exist an abstract trajectory $r = r_{x,y} \in \mathcal{R}$ and elements $t, \tau \in \mathfrak{D}(r)$ such, that $x = r(t)$, $y = r(\tau)$ and $t \leq \tau$. Also we define the following mapping $\varphi_{\mathcal{R}} : \mathbf{T} \mapsto 2^M$:

$$\varphi_{\mathcal{R}}(t) = \bigcup_{r \in \mathcal{R}, t \in \mathfrak{D}(r)} \{r(t)\} = \{r(t) \mid r \in \mathcal{R}, t \in \mathfrak{D}(r)\}.$$

In particular, $\varphi_{\mathcal{R}}(t) = \emptyset$ in the case, where there not exist a trajectory $r \in \mathcal{R}$ such, that $t \in \mathfrak{D}(r)$.

It is not hard to verify, that the pair $\mathcal{M} = \left(M, \overset{\mathcal{R}}{\leftarrow}\right)$ is an oriented set and the mapping $\varphi_{\mathcal{R}}$ is a time on \mathcal{M} . Therefore, the triple:

$$\mathcal{P} = (\mathcal{M}, \mathbb{T}, \varphi_{\mathcal{R}}) = \left(\left(M, \overset{\mathcal{R}}{\leftarrow}\right), (\mathbf{T}, \leq), \varphi_{\mathcal{R}}\right)$$

is a primitive changeable set. And it is not hard to see, that this primitive changeable set satisfies the conditions 1),2),3) of this theorem.

Inversely, if a primitive changeable set \mathcal{P}_1 satisfies the conditions 1),2),3) of this theorem, then from the first condition it follows, that $\mathfrak{Bs}(\mathcal{P}_1) = M$, $\mathbf{Tm}(\mathcal{P}_1) = \mathbf{T}$, $\leq_{\mathcal{P}_1} = \leq$. And the second and third conditions imply the equalities $\leftarrow_{\mathcal{P}_1} = \leftarrow_{\mathcal{R}}$, $\psi_{\mathcal{P}_1} = \varphi_{\mathcal{R}}$. Thus,

$$\mathcal{P}_1 = \left(\left(\mathfrak{Bs}(\mathcal{P}_1), \leftarrow_{\mathcal{P}_1} \right), (\mathbf{Tm}(\mathcal{P}_1), \leq_{\mathcal{P}_1}), \psi_{\mathcal{P}_1} \right) = \left(\left(M, \leftarrow_{\mathcal{R}} \right), (\mathbf{T}, \leq), \varphi_{\mathcal{R}} \right) = \mathcal{P}.$$

□

Definition 6.2. Let \mathcal{R} be any system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to M . The primitive changeable set \mathcal{P} , which satisfies the conditions 1),2),3) of the theorem 6.1, will be named a primitive changeable set, **generated by the system of abstract trajectories \mathcal{R}** , and it will be denoted by $\mathcal{Atp}(\mathbb{T}, \mathcal{R})$:

$$\mathcal{Atp}(\mathbb{T}, \mathcal{R}) := \mathcal{P}.$$

Thus, systems of abstract trajectories provide the simple tool for creation of primitive changeable sets.

7 Elementary-time States and Basic Changeable Sets

7.1 Elementary-time States of Primitive Changeable Sets and their Properties

Definition 7.1. Let \mathcal{P} be a primitive changeable set. Any pair (t, x) ($x \in \mathfrak{Bs}(\mathcal{P})$, $t \in \mathbf{Tm}(\mathcal{P})$) such, that $x \in \psi(t)$, will be named an **elementary-time state**.

The set of all elementary-time states of \mathcal{P} will be denoted by $\mathbb{Bs}(\mathcal{P})$:

$$\mathbb{Bs}(\mathcal{P}) := \{ \omega \mid \omega = (t, x), \text{ where } t \in \mathbf{Tm}(\mathcal{P}), x \in \psi(t) \}.$$

For any elementary-time state $\omega = (t, x) \in \mathbb{Bs}(\mathcal{P})$ we introduce the following denotations:

$$\mathbf{bs}(\omega) := x, \quad \mathbf{tm}(\omega) := t.$$

Definition 7.2. We will say, that an elementary-time state $\omega_2 \in \mathbb{Bs}(\mathcal{P})$ is **formally sequential** to an elementary-time state $\omega_1 \in \mathbb{Bs}(\mathcal{P})$ if and only if $\omega_1 = \omega_2$ or $\mathbf{bs}(\omega_2) \leftarrow_{\mathcal{P}} \mathbf{bs}(\omega_1)$ and $\mathbf{tm}(\omega_1) <_{\mathcal{P}} \mathbf{tm}(\omega_2)$. For this case we will use the denotation:

$$\omega_2 \leftarrow_{\mathcal{P}} (\mathbf{f}) \omega_1.$$

In the case, where the primitive changeable set \mathcal{P} , in question is known, in the denotation $\omega_2 \leftarrow_{\mathcal{P}} (\mathbf{f}) \omega_1$ the symbol \mathcal{P} will be omitted, and the notation $\omega_2 \leftarrow (\mathbf{f}) \omega_1$ will be used instead.

Assertion 7.1. 1) If $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$ and $\omega_2 \leftarrow (\mathbf{f}) \omega_1$, then $\mathbf{tm}(\omega_1) \leq \mathbf{tm}(\omega_2)$. If, in addition, $\omega_1 \neq \omega_2$, then $\mathbf{tm}(\omega_1) < \mathbf{tm}(\omega_2)$.

2) The relation $\leftarrow (\mathbf{f}) = \leftarrow_{\mathcal{P}} (\mathbf{f})$ is asymmetric on the set $\mathbb{Bs}(\mathcal{P})$, that is if $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$, $\omega_2 \leftarrow (\mathbf{f}) \omega_1$ and $\omega_1 \leftarrow (\mathbf{f}) \omega_2$, then $\omega_1 = \omega_2$.

Proof. The first statement follows by a trivial way from the definition 7.2, and the second statement derives from the first. □

Definition 7.3. The oriented set \mathcal{M} is named **anti-cyclical** if for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ the conditions $x \leftarrow y$ and $y \leftarrow x$ involve the equality $x = y$.

Assertion 7.2. *Let \mathcal{P} be a primitive changeable set. Then:*

- 1) *The pair $\mathcal{Q} = \left(\mathbb{B}\mathfrak{s}(\mathcal{P}), \leftarrow_{\mathcal{P}}(\mathfrak{f}) \right) = (\mathbb{B}\mathfrak{s}(\mathcal{P}), \leftarrow(\mathfrak{f}))$ is an anti-cyclical oriented set.*
- 2) *The mapping:*

$$\tilde{\psi}(t) = \tilde{\psi}_{\mathcal{P}}(t) := \{\omega \in \mathbb{B}\mathfrak{s}(\mathcal{P}) \mid \mathbf{tm}(\omega) = t\} \in 2^{\mathbb{B}\mathfrak{s}(\mathcal{P})}, \quad t \in \mathbf{Tm}(\mathcal{P}) \quad (9)$$

is a monotone time on \mathcal{Q} .

- 3) *For $t_1 \neq t_2$ we have $\tilde{\psi}(t_1) \cap \tilde{\psi}(t_2) = \emptyset$.*
- 4) *If, in addition, $\psi(t) \neq \emptyset, t \in \mathbf{Tm}(\mathcal{P})$, then the time ψ is strictly monotone.*

Proof. 1) The first statement of the assertion 7.2 follows from the definition 7.2 and second statement of the assertion 7.1.

2) 2.1) Let $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{P})$. Then, by (9), $\omega \in \tilde{\psi}(t)$, where $t = \mathbf{tm}(\omega)$.

2.2) Let $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{P})$, $\omega_2 \leftarrow(\mathfrak{f}) \omega_1$ and $\omega_1 \neq \omega_2$. According to (9), for $t_1 = \mathbf{tm}(\omega_1)$, $t_2 = \mathbf{tm}(\omega_2)$ we obtain:

$$\omega_1 \in \tilde{\psi}(t_1), \quad \omega_2 \in \tilde{\psi}(t_2).$$

Since $\omega_2 \leftarrow(\mathfrak{f}) \omega_1$ and $\omega_2 \neq \omega_1$, then, by assertion 7.1 (statement 1), $t_1 < t_2$.

From 2.1), 2.2) it follows, that $\tilde{\psi}$ is a time on \mathcal{Q} .

2.3) Let $\omega_1 \in \tilde{\psi}(t_1)$, $\omega_2 \in \tilde{\psi}(t_2)$, $\omega_2 \leftarrow(\mathfrak{f}) \omega_1$ and $\omega_1 \not\leftarrow(\mathfrak{f}) \omega_2$. Then, by definition of time $\tilde{\psi}$ (9), $\mathbf{tm}(\omega_1) = t_1$, $\mathbf{tm}(\omega_2) = t_2$. Therefore, by assertion 7.1, statement 1, $t_1 < t_2$. Thus, the time $\tilde{\psi}$ is monotone.

3) Let $t_1, t_2 \in \mathbf{Tm}(\mathcal{P})$. Suppose, that $\tilde{\psi}(t_1) \cap \tilde{\psi}(t_2) \neq \emptyset$. Then there exists an elementary-time state $\omega \in \tilde{\psi}(t_1) \cap \tilde{\psi}(t_2)$. Hence, by (9), we obtain $t_1 = \mathbf{tm}(\omega) = t_2$.

4) Assume, that, in addition, $\psi(t) \neq \emptyset, t \in \mathbf{Tm}(\mathcal{P})$. Then for an arbitrary $t \in \mathbf{Tm}(\mathcal{P})$ there exists an elementary state $x_t \in \mathfrak{B}\mathfrak{s}(\mathcal{P})$ such, that $x_t \in \psi(t)$. Consequently, the elementary-time state $\omega_t = (t, x_t) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ satisfies the condition $\mathbf{tm}(\omega_t) = t$, that is $\omega_t \in \tilde{\psi}(t)$. Thus, $\tilde{\psi}(t) \neq \emptyset, t \in \mathbf{Tm}(\mathcal{P})$. Hence, taking into account the statement 3) of this assertion, we obtain, $\tilde{\psi}(t_1) \neq \tilde{\psi}(t_2)$ for $t_1, t_2 \in \mathbf{Tm}(\mathcal{P})$, $t_1 \neq t_2$. Consequently, the time $\tilde{\psi}$ is incessant, and, taking into account the statement 2) of this assertion, we conclude, that the time $\tilde{\psi}$ is strictly monotone. \square

7.2 Base of Elementary Processes and Basic Changeable Sets

As it had been proved in the assertion 7.2, for any primitive changeable set \mathcal{P} the pair $(\mathbb{B}\mathfrak{s}(\mathcal{P}), \leftarrow(\mathfrak{f}))$ is an oriented set, in which $\leftarrow(\mathfrak{f})$ is the directing relation of changes. But, it turns out, that in the reality, the relation $\leftarrow(\mathfrak{f})$ may generate such “transformations” of elementary-time states, which never took place in the real physical system. To illustrate this fact, we consider the following example.

Example 7.1. Let us consider the system of abstract trajectories, which describes the uniform linear motion of the system of identical material points, evenly distributed on the straight trajectory of their own motion. The identity of the material points assumes, that all characteristics of these points in a some time moment can be reduced to only their coordinates. This means, that a material point, which has a certain coordinates at a some time moment is completely mathematically identical to the one point that have the same coordinates in another time. This system of material points can be described by the following system of abstract trajectories from \mathbb{R} to \mathbb{R} :

$$\mathcal{R} = \{r_\alpha \mid \alpha \in \mathbb{R}\}, \quad \text{where}$$

$$r_\alpha(t) := t + \alpha, \quad t \in \mathbb{R}, \quad \alpha \in \mathbb{R} \quad (\mathfrak{D}(r_\alpha) = \mathbb{R}, \alpha \in \mathbb{R}).$$

Denote:

$$\mathcal{P} := \mathcal{Atp}((\mathbb{R}, \leq), \mathcal{R}),$$

where “ \leq ” is the standard linear order relation on the real numbers. By the definition 6.2 and condition 1) of the theorem 6.1, $\mathfrak{Bs}(\mathcal{P}) = \mathbf{Tm}(\mathcal{P}) = \mathbb{R}$. We are aim to prove, that for the elements $x_1, x_2 \in \mathfrak{Bs}(\mathcal{P}) = \mathbb{R}$ the condition $x_2 \leftarrow x_1$ is equivalent to the inequality $x_1 \leq x_2$. Indeed, in the case $x_1 \leq x_2$ for $t_1 = x_1, t_2 = x_2$ we obtain $x_1 = r_0(t_1), x_2 = r_0(t_2)$, where $t_1 \leq t_2$. Therefore, by the condition 2) of the theorem 6.1, we obtain $x_2 \leftarrow x_1$. Inversely, if $x_2 \leftarrow x_1$, then, by condition 2) of the theorem 6.1, there exist numbers $\alpha, t_1, t_2 \in \mathbb{R}$ such, that $t_1 \leq t_2, x_1 = r_\alpha(t_1), x_2 = r_\alpha(t_2)$, that is $x_1 = t_1 + \alpha, x_2 = t_2 + \alpha$, where $t_1 \leq t_2$. Hence, $x_1 \leq x_2$.

The next aim is to prove, that $\mathbb{Bs}(\mathcal{P}) = \mathbb{R} \times \mathbb{R}$. Since $\mathfrak{Bs}(\mathcal{P}) = \mathbf{Tm}(\mathcal{P}) = \mathbb{R}$, we have $\mathbb{Bs}(\mathcal{P}) \subseteq \mathbb{R} \times \mathbb{R}$. Thus, it remains to prove, the inverse inclusion. Let $\omega = (\tau, x) \in \mathbb{R} \times \mathbb{R}$. Denote $\alpha_\omega := x - \tau$. Then $r_{\alpha_\omega}(\tau) = \tau + (x - \tau) = x$. Therefore, by condition 3) of the theorem 6.1, $x \in \psi_{\mathcal{P}}(\tau)$. This means, that $\omega = (\tau, x) \in \mathbb{Bs}(\mathcal{P})$. The equality $\mathbb{Bs}(\mathcal{P}) = \mathbb{R} \times \mathbb{R}$ has been proved.

By definition 7.2 of formally sequential elementary-time states, for $\omega_1 = (t_1, x_1), \omega_2 = (t_2, x_2) \in \mathbb{Bs}(\mathcal{P})$ the condition $\omega_2 \leftarrow (f) \omega_1$ is performed if and only if $\omega_1 = \omega_2$ or $t_1 < t_2$ and $x_1 \leq x_2$. Hence, if we choose any elementary-time states $\omega_1 = (t_1, x_1), \omega_2 = (t_2, x_2) \in \mathbb{Bs}(\mathcal{P}) = \mathbb{R} \times \mathbb{R}$, satisfying $t_1 < t_2$ and $x_1 \leq x_2$, we obtain $\omega_2 \leftarrow (f) \omega_1$. But in the case $x_1 - t_1 \neq x_2 - t_2$ there not exist an abstract trajectory $r_\alpha \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in r_\alpha$. This means, that in this model of real physical process, the elementary-time state ω_2 may not be the result of transformations of the elementary-time state ω_1 . Thus, in this example, the relation $\leftarrow (f)$ generates infinitely many “parasitic transformation relations”, which never took place in the reality.

The above example shows, that to adequate describe real physical process, the directing relation of changes should be defined not only on the set of set of elementary states $\mathfrak{Bs}(\mathcal{P})$, but, also, on the set of elementary-time states $\mathbb{Bs}(\mathcal{P})$ of a primitive changeable set \mathcal{P} . Indeed, let us consider the primitive changeable set $\mathcal{P} := \mathcal{Atp}(\mathcal{R})$ from the example 7.1. For $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$ we can put $\omega_2 \leftarrow \omega_1$ if and only if $\mathbf{tm}(\omega_1) \leq \mathbf{tm}(\omega_2)$ and there exist an abstract trajectory $r_\alpha \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in r_\alpha$ (that is such, that $\mathbf{bs}(\omega_1) = r_\alpha(\mathbf{tm}(\omega_1)), \mathbf{bs}(\omega_2) = r_\alpha(\mathbf{tm}(\omega_2))$). Thus, we obtain the relation “ \leftarrow ”, which reflects only such transformations of the elementary-time states, which actually take place in the reality.

Definition 7.4. Let \mathcal{P} be a primitive changeable set.

1. Relation \leftarrow on $\mathbb{Bs}(\mathcal{P})$ is named **base of elementary processes** if and only if:

- (1) $\forall \omega \in \mathbb{Bs}(\mathcal{P}) \quad \omega \leftarrow \omega$.
- (2) If $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$ and $\omega_2 \leftarrow \omega_1$, then $\omega_2 \leftarrow (f) \omega_1$ (ie $\leftarrow \subseteq \leftarrow (f)$).
- (3) For arbitrary $x_1, x_2 \in \mathfrak{Bs}(\mathcal{P})$ such, that $x_2 \leftarrow x_1$ there exist $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$ such, that $\mathbf{bs}(\omega_1) = x_1, \mathbf{bs}(\omega_2) = x_2$ and $\omega_2 \leftarrow \omega_1$.

2. In the case, where \leftarrow is the base of elementary processes on the primitive changeable set \mathcal{P} , the pair:

$$\mathcal{B} = (\mathcal{P}, \leftarrow)$$

will be referred to as **basic changeable set**.

7.3 Remarks on the Denotations

For further basic changeable sets will be denoted by large calligraphy symbols.

Let $\mathcal{B} = (\mathcal{P}, \leftarrow)$ be a basic changeable set. We introduce the following denotations:

$$\begin{aligned}
\mathfrak{Bs}(\mathcal{B}) &:= \mathfrak{Bs}(\mathcal{P}); & \mathbb{Bs}(\mathcal{B}) &:= \mathbb{Bs}(\mathcal{P}); & \leftarrow_{\mathcal{B}} &:= \leftarrow_{\mathcal{P}}; \\
\leftarrow_{\mathcal{B}}(f) &:= \leftarrow_{\mathcal{P}}(f); & \mathbf{Tm}(\mathcal{B}) &:= \mathbf{Tm}(\mathcal{P}); & \leq_{\mathcal{B}} &:= \leq_{\mathcal{P}}; \\
<_{\mathcal{B}} &:= <_{\mathcal{P}}; & \geq_{\mathcal{B}} &:= \geq_{\mathcal{P}}; & >_{\mathcal{B}} &:= >_{\mathcal{P}}; \\
\psi_{\mathcal{B}} &:= \psi_{\mathcal{P}}.
\end{aligned}$$

Also for elementary-time states $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{B})$ we will use the denotation $\omega_2 \leftarrow_{\mathcal{B}} \omega_1$ instead of the denotation $\omega_2 \leftarrow \omega_1$.

In the case, where the basic changeable set \mathcal{P} , is clear in the denotations $\leftarrow_{\mathcal{B}}, \leftarrow_{\mathcal{B}}(f), \leq_{\mathcal{B}}, <_{\mathcal{B}}, \geq_{\mathcal{B}}, >_{\mathcal{B}}, \psi_{\mathcal{B}}$ the symbol \mathcal{B} will be omitted, and the denotations $\leftarrow, \leftarrow(f), \leq, <, \geq, >, \psi$ will be used instead.

The next properties of basic changeable sets follow from the definitions 7.4 and 7.2 (in the properties 1-5 the symbol \mathcal{B} means a basic changeable set):

Properties 7.1.

1. The pair $\mathcal{B}_0 = (\mathfrak{Bs}(\mathcal{B}), \leftarrow)$ is an oriented set.
2. The mapping $\psi = \psi_{\mathcal{B}}$ is a time on $\mathcal{B}_0 = (\mathfrak{Bs}(\mathcal{B}), \leftarrow)$.
3. $\omega \leftarrow \omega$ for any elementary-time state $\omega \in \mathbb{Bs}(\mathcal{B})$.
4. If $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{B})$ and $\omega_2 \leftarrow \omega_1$, then $\omega_2 \leftarrow(f) \omega_1$, and therefore $\mathbf{bs}(\omega_2) \leftarrow \mathbf{bs}(\omega_1)$ and $\mathbf{tm}(\omega_1) \leq \mathbf{tm}(\omega_2)$.
5. For arbitrary $x_1, x_2 \in \mathfrak{Bs}(\mathcal{B})$ the condition $x_2 \leftarrow x_1$ holds if and only if there exist elementary-time states $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{B})$ such, that $\mathbf{bs}(\omega_1) = x_1$, $\mathbf{bs}(\omega_2) = x_2$ and $\omega_2 \leftarrow \omega_1$.
6. $\mathfrak{Bs}(\mathcal{B}) = \{\mathbf{bs}(\omega) \mid \omega \in \mathbb{Bs}(\mathcal{B})\}$.

7.4 Examples of Basic Changeable Sets

Example 7.2. Let \mathcal{P} be any primitive changeable set. Then the relation $\leftarrow(f) = \leftarrow_{\mathcal{P}}(f)$ is base of elementary processes on \mathcal{P} . Indeed, the conditions (1) and (2) of the definition 7.4 for the relation $\leftarrow(f)$ are fulfilled by a trivial way. To verify the condition (3) we consider arbitrary $x_1, x_2 \in \mathfrak{Bs}(\mathcal{P})$ such, that $x_2 \leftarrow x_1$. In the case $x_1 = x_2$ by the time definition 3.1, there exist a time point $t_1 \in \mathbf{Tm}(\mathcal{P})$ such, that $x_1 \in \psi(t_1)$. Hence, for $\omega_1 = \omega_2 = (t_1, x_1) \in \mathbb{Bs}(\mathcal{P})$ we obtain $\mathbf{bs}(\omega_1) = \mathbf{bs}(\omega_2) = x_1 = x_2$ and $\omega_2 \leftarrow(f) \omega_1$. Thus, in the case $x_1 = x_2$ the condition (3) of the definition 7.4 is satisfied. In the case $x_1 \neq x_2$, by definition 3.1, there exist time points $t_1, t_2 \in \mathbf{Tm}(\mathcal{P})$ such, that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$. Hence, for $\omega_1 = (t_1, x_1)$, $\omega_2 = (t_2, x_2) \in \mathbb{Bs}(\mathcal{P})$, we obtain $\mathbf{bs}(\omega_1) = x_1$, $\mathbf{bs}(\omega_2) = x_2$ and $\omega_2 \leftarrow(f) \omega_1$. Thus, in the case $x_1 \neq x_2$ the condition (3) of the definition 7.4 also is satisfied.

Therefore any primitive changeable set can be interpreted as basic changeable set $\mathcal{P}_{(f)} = (\mathcal{P}, \leftarrow(f))$ in with the relation $\leftarrow(f)$ is the base of elementary processes.

Example 7.3. Let \mathcal{R} be any system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to M . Denote:

$$\mathcal{P} := \mathcal{Atp}(\mathbb{T}, \mathcal{R}).$$

By theorem 6.1, $\mathfrak{Bs}(\mathcal{P}) = M$, $\mathbf{Tm}(\mathcal{P}) = \mathbf{T}$. Moreover, by third statement of the theorem for $(t, x) \in M \times \mathbf{T}$ the condition $(t, x) \in \mathbb{Bs}(\mathcal{P})$ holds if and only if there exist an abstract trajectory $r = r_{t,x} \in \mathcal{R}$ such, that $t \in \mathfrak{D}(r)$ and $x = r(t)$, ie such, that $\omega = (t, x) \in r$. Thus,

$$\mathbb{Bs}(\mathcal{P}) = \bigcup_{r \in \mathcal{R}} r. \quad (10)$$

Then, for $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ we put $\omega_2 \leftarrow [\mathcal{R}] \omega_1$ if and only if $\mathbf{tm}(\omega_1) \leq \mathbf{tm}(\omega_2)$ and there exists an abstract trajectory $r \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in \mathcal{R}$ (ie such, that $\mathbf{bs}(\omega_1) = r(\mathbf{tm}(\omega_1))$, $\mathbf{bs}(\omega_2) = r(\mathbf{tm}(\omega_2))$). We are going to prove, that the relation $\leftarrow [\mathcal{R}]$ provides base of elementary processes.

(a) Let $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{P})$. Then, by (10), there exist an abstract trajectory $r \in \mathcal{R}$ such, that $\omega \in r$. Hence, by definition of the relation “ $\leftarrow [\mathcal{R}]$ ”, we have $\omega \leftarrow [\mathcal{R}] \omega$.

(b) Let $\omega_1 = (t_1, x_1)$, $\omega_2 = (t_2, x_2) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ and $\omega_2 \leftarrow [\mathcal{R}] \omega_1$. Then, from definition of the relation “ $\leftarrow [\mathcal{R}]$ ”, it follows, that $t_1 \leq t_2$ and there exists an abstract trajectory $r \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in \mathcal{R}$ (ie such, that $x_1 = r(t_1)$, $x_2 = r(t_2)$). Consequently, by statement 2) of the theorem 6.1, $x_2 \xleftarrow{\mathcal{Atp}(\mathcal{R})} x_1$. Therefore, in the case $t_1 \neq t_2$ we have $t_1 < t_2$ and $x_2 \leftarrow x_1$, besides in the case $t_1 = t_2$ we obtain $x_1 = r(t_1) = r(t_2) = x_2$, that is $\omega_1 = \omega_2$. But, in the both cases the correlation $\omega_2 \leftarrow (f) \omega_1$ is true.

(c) Let $x_1, x_2 \in \mathfrak{B}\mathfrak{s}(\mathcal{P})$, $x_2 \leftarrow x_1$ (ie $x_2 \xleftarrow{\mathcal{Atp}(\mathcal{R})} x_1$). Then, by statement 2) of the theorem 6.1, there exists an abstract trajectory $r \in \mathcal{R}$ such, that $x_1 = r(t_1)$, $x_2 = r(t_2)$ for some $t_1, t_2 \in \mathbf{Tm}(\mathcal{P})$ such, that $t_1 \leq t_2$. Denote:

$$\omega_i := (t_i, x_i), \quad i \in \{1, 2\}.$$

Then, $\omega_1, \omega_2 \in r \subseteq \bigcup_{\rho \in \mathcal{R}} \rho = \mathbb{B}\mathfrak{s}(\mathcal{P})$, $\mathbf{bs}(\omega_i) = x_i$ ($i \in \{1, 2\}$) and, by definition of the relation “ $\leftarrow [\mathcal{R}]$ ”, $\omega_2 \leftarrow [\mathcal{R}] \omega_1$.

From the items (a)-(c) it follows, that the relation $\leftarrow [\mathcal{R}]$ is base of elementary processes on $\mathcal{P} = \mathcal{Atp}(\mathbb{T}, \mathcal{R})$. Thus, the pair:

$$\mathcal{At}(\mathbb{T}, \mathcal{R}) = (\mathcal{P}, \leftarrow [\mathcal{R}]) = (\mathcal{Atp}(\mathbb{T}, \mathcal{R}), \leftarrow [\mathcal{R}])$$

is a basic changeable set.

From the properties 7.1(5,6) it follows, that if for a some basic changeable set \mathcal{B} we know $\mathbf{Tm}(\mathcal{B})$, $\leq_{\mathcal{B}}$, $\mathbb{B}\mathfrak{s}(\mathcal{B})$ and base of elementary processes $\xleftarrow{\mathcal{B}}$ on $\mathbb{B}\mathfrak{s}(\mathcal{B})$, then we can recover the set $\mathfrak{B}\mathfrak{s}(\mathcal{B})$, the directing relation of changes $\xleftarrow{\mathcal{B}}$ on $\mathfrak{B}\mathfrak{s}(\mathcal{B})$ and the time $\psi_{\mathcal{B}}(t)$ (using the formula $\psi_{\mathcal{B}}(t) = \{x \in \mathfrak{B}\mathfrak{s}(\mathcal{B}) \mid (t, x) \in \mathbb{B}\mathfrak{s}(\mathcal{B})\}$, $t \in \mathbf{Tm}(\mathcal{B})$), and thus, we can recover the whole basic changeable set \mathcal{B} . Hence from the last example it follows the next theorem.

Theorem 7.1. *Let \mathcal{R} be a system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to M . Then there exists a unique basic changeable set $\mathcal{B} = \mathcal{At}(\mathbb{T}, \mathcal{R})$, such, that:*

- 1) $(\mathbf{Tm}(\mathcal{At}(\mathbb{T}, \mathcal{R})), \leq_{\mathcal{At}(\mathbb{T}, \mathcal{R})}) = \mathbb{T}$;
- 2) $\mathbb{B}\mathfrak{s}(\mathcal{At}(\mathbb{T}, \mathcal{R})) = \bigcup_{r \in \mathcal{R}} r$;
- 3) *For arbitrary $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{At}(\mathbb{T}, \mathcal{R}))$ the condition $\omega_2 \xleftarrow{\mathcal{At}(\mathbb{T}, \mathcal{R})} \omega_1$ is satisfied if and only if $\mathbf{tm}(\omega_1) \leq \mathbf{tm}(\omega_2)$ and there exist an abstract trajectory $r \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in r$.*

Remark 7.1. 1. Since the construction of the basic changeable set $\mathcal{At}(\mathbb{T}, \mathcal{R})$ is based on the primitive changeable set $\mathcal{Atp}(\mathbb{T}, \mathcal{R})$, for any basic changeable set of kind $\mathcal{B} = \mathcal{At}(\mathbb{T}, \mathcal{R})$ the statements, formulated in the items 2), 3) of the theorem 6.1 remain true (with replacement the character \mathcal{P} by \mathcal{B} or by $\mathcal{At}(\mathbb{T}, \mathcal{R})$).

2. In the case, when the linear ordered set \mathbb{T} is given in advance, we will use the denotation $\mathcal{At}(\mathcal{R})$ instead of $\mathcal{At}(\mathbb{T}, \mathcal{R})$.

8 Chains in the Set of Elementary-time States. Fate Lines and their Properties

Using the definition of basic changeable sets as well as the assertions 7.2 and 7.1 (item 2) we obtain the following assertion.

Assertion 8.1. *Let \mathcal{B} be a basic changeable set. Then:*

- 1) *The pair $\mathcal{Q} = (\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow_{\mathcal{B}}) = (\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is an anti-cyclical oriented set.*
- 2) *The mapping*

$$\tilde{\psi}(t) = \tilde{\psi}_{\mathcal{B}}(t) := \{\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}) \mid \mathbf{tm}(\omega) = t\} \in 2^{\mathbb{B}\mathfrak{s}(\mathcal{B})}, \quad t \in \mathbf{Tm}(\mathcal{B}) \quad (11)$$

is a monotone time on \mathcal{Q} .

- 3) *If, in addition, $\psi(t) \neq \emptyset$, $t \in \mathbf{Tm}(\mathcal{P})$, then the time ψ is strictly monotone.*

According to the assertion 8.1, for any basic changeable set \mathcal{B} the pair $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is (anti-cyclical) oriented set. As in an arbitrary oriented set, in $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ we may introduce transitive sets and chains. From anti-cyclicity of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ it follows the following assertion.

Assertion 8.2. *Let \mathcal{B} be a basic changeable set.*

- 1) *Any transitive subset $\mathcal{N} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is a (partially) ordered set (relatively the relation \leftarrow).*
- 2) *Any chain $\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is a linearly ordered set (relatively the relation \leftarrow).*

Definition 8.1. *Let \mathcal{B} be a basic changeable set.*

- 1) *Any maximum chain $\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ will be named a **fate line** of \mathcal{B} . The set of all fate lines of \mathcal{B} will be denoted by $\mathbb{Ld}(\mathcal{B})$:*

$$\mathbb{Ld}(\mathcal{B}) = \{\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}) \mid \mathcal{L} \text{ is a fate line of } \mathcal{B}\}.$$

- 2) *Any fate line, which contains an elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ will be named the (**eigen**) fate line of elementary-time state ω (in \mathcal{B}).*
- 3) *A fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$ will be named the (**eigen**) fate line of the elementary state $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ if and only if there exists the elementary-time state $\omega_x \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\mathbf{bs}(\omega_x) = x$ and \mathcal{L} is eigen fate line of ω_x .*

It is clear that, in the general case, an elementary (elementary-time) state may have many fate lines.

We will say, that elementary (elementary-time) states $x_1, x_2 \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$, ($\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$) are **united by fate** if and only if there exist at least one fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$, which is eigen fate line of both states x_1, x_2 (ω_1, ω_2).

Assertion 8.3. 1) *Any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ must have at least one eigen fate line.*

- 2) *For elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ to be united by fate it is necessary and sufficient satisfaction one of the following conditions:*

$$\omega_2 \leftarrow \omega_1 \quad \text{or} \quad \omega_1 \leftarrow \omega_2. \quad (12)$$

Proof. 1) The first statement of this assertion follows from the corollary 2.2.

2) 2.a) Suppose, that for the elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ there exist a common fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$ ($\omega_1, \omega_2 \in \mathcal{L}$). Then, by assertion 8.2, item 2, the pair $(\mathcal{L}, \leftarrow)$ is a linearly ordered set. Thus at least one of the conditions (12) must be fulfilled.

2.b) Let, $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ and $\omega_2 \leftarrow \omega_1$. Then, by corollary 2.2, there exist a maximum chain (fate line) $\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ ($\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$) such, that $\omega_1, \omega_2 \in \mathcal{L}$. \square

Assertion 8.4. 1) Any elementary state $x \in \mathfrak{Bs}(\mathcal{B})$ must have at least one eigen fate line.

2) For elementary states $x, y \in \mathfrak{Bs}(\mathcal{B})$ to be united by fate it is necessary and sufficient satisfaction one of the following conditions:

$$y \leftarrow x \quad \text{or} \quad x \leftarrow y. \quad (13)$$

Proof. 1) Let $x \in \mathfrak{Bs}(\mathcal{B})$. Then, by the definition of time, there exist a time point $t \in \mathbf{Tm}(\mathcal{B})$ such, that $x \in \psi(t)$. By assertion 8.3, the elementary-time state $\omega_x = (t, x) \in \mathbb{Bs}(\mathcal{B})$ must have an eigen fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$. This fate line \mathcal{L} must be eigen fate line of elementary state x .

2) 2.a) Let $x, y \in \mathfrak{Bs}(\mathcal{B})$, $y \leftarrow x$. Then, by the property 7.1(5) (see properties 7.1), there exist elementary-time states $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$ such, that $\mathbf{bs}(\omega_1) = x$, $\mathbf{bs}(\omega_2) = y$ and $\omega_2 \leftarrow \omega_1$. By assertion 8.3, there exist a common fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$ for the elementary-time states ω_1, ω_2 (ie $\omega_1, \omega_2 \in \mathcal{L}$). By definition 8.1, this fate line \mathcal{L} must be eigen fate line of both elementary states x and y .

2.b) Suppose, that for the elementary states $x, y \in \mathfrak{Bs}(\mathcal{B})$ there exist a common eigen fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$. Then, there exist elementary-time states $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{B})$, such, that $\mathbf{bs}(\omega_1) = x$, $\mathbf{bs}(\omega_2) = y$ and $\omega_1, \omega_2 \in \mathcal{L}$. Hence, by assertion 8.3, statement 2), one of the conditions $\omega_2 \leftarrow \omega_1$ or $\omega_1 \leftarrow \omega_2$ must be satisfied. Then, by the property 7.1(4), at least one of the conditions (13) must be fulfilled. \square

As it was shown in the theorem 7.1, any system of abstract trajectories, defined on some linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, generates the basic changeable set $\mathcal{At}(\mathbb{T}, \mathcal{R})$. The next aim is to show, that any basic changeable set \mathcal{B} can be represented in the form $\mathcal{B} = \mathcal{At}(\mathbb{T}, \mathcal{R})$, where \mathcal{R} is some system of abstract trajectories, defined on some linearly ordered set \mathbb{T} .

Definition 8.2. Let \mathcal{R} be a system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to M .

1. Trajectory $r \in \mathcal{R}$ will be named a **maximum trajectory** (relatively the \mathcal{R}) if and only if there not exist any trajectory $\rho \in \mathcal{R}$ ($\rho \neq r$) such, that $\mathfrak{D}(r) \subset \mathfrak{D}(\rho)$ and $r(t) = \rho(t)$ $t \in \mathfrak{D}(r)$ (that is such, that $r \subset \rho$).
2. The system of abstract trajectories \mathcal{R} will be referred to as the **system of maximum trajectories** if and only if any trajectory $r \in \mathcal{R}$ is maximum trajectory (relatively the \mathcal{R}).

Further, for any basic changeable set \mathcal{B} we will use the denotation:

$$\mathbf{Tm}(\mathcal{B}) := (\mathbf{Tm}(\mathcal{B}), \leq_{\mathcal{B}}).$$

Assertion 8.5. Let \mathcal{B} be a basic changeable set. Then:

- 1) Any chain $\mathcal{L} \subseteq \mathbb{Bs}(\mathcal{B})$ of the oriented set $(\mathbb{Bs}(\mathcal{B}), \leftarrow)$ is an abstract trajectory from $\mathbf{Tm}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$.
- 2) The set $\mathbb{Ll}(\mathcal{B})$ of all chains of the oriented set $(\mathbb{Bs}(\mathcal{B}), \leftarrow)$ is a system of abstract trajectories from $\mathbf{Tm}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$.
- 3) Any fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B}) \subseteq \mathbb{Ll}(\mathcal{B})$ is a maximum trajectory (relatively the system of abstract trajectories $\mathbb{Ll}(\mathcal{B})$).
- 4) The set $\mathbb{Ld}(\mathcal{B})$ is a system of maximum trajectories (from $\mathbf{Tm}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$).

Proof. 1) Let $\mathcal{L} \subseteq \mathbb{Bs}(\mathcal{B})$ be a chain of the oriented set $(\mathbb{Bs}(\mathcal{B}), \leftarrow)$. Since $\mathbb{Bs}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{Bs}(\mathcal{B})$ and $\mathcal{L} \subseteq \mathbb{Bs}(\mathcal{B})$, then \mathcal{L} is a binary relation from the set $\mathbf{Tm}(\mathcal{B})$ to the set $\mathfrak{Bs}(\mathcal{B})$. Thus, to make sure that \mathcal{L} is an abstract trajectory from $\mathbf{Tm}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$, it is sufficient to prove, that this relation \mathcal{L} is a function from $\mathbf{Tm}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$. Suppose contrary. Then there exist elementary-time states $\omega_1, \omega_2 \in \mathcal{L}$ of kind $\omega_1 = (t, x_1)$, $\omega_2 = (t, x_2)$, where $x_1 \neq x_2$. Since \mathcal{L} is a chain, one of the conditions $\omega_2 \leftarrow \omega_1$ or $\omega_1 \leftarrow \omega_2$ must be satisfied. Assume, that $\omega_2 \leftarrow \omega_1$. Then, by the property 7.1(4) (see properties 7.1), $\omega_2 \leftarrow (f)\omega_1$. Hence, taking into account, that $\omega_2 \neq \omega_1$, by assertion 7.1 (item 1), we obtain $t < t$, which is impossible. Similarly

the assumption $\omega_1 \leftarrow \omega_2$ also leads to contradiction. The obtained contradiction proves that the chain \mathcal{L} is a function.

Taking into account, that, according to the proved above, any chain \mathcal{L} of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is an abstract trajectory, we may use the notations $\mathfrak{D}(\mathcal{L})$ for the domain of \mathcal{L} and $x = \mathcal{L}(t)$ (where $t \in \mathfrak{D}(\mathcal{L})$) to indicate the fact that $(t, x) \in \mathcal{L}$.

2) Let $\mathbb{Ll}(\mathcal{B})$ be the set of all chains of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$. Chose any elementary state $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$. By the time definition, there exist a time point $t \in \mathbf{Tm}(\mathcal{B})$ such, that $x \in \psi(t)$. By assertion 2.1, item 2, the singleton set $\mathcal{L}_x = \{(t, x)\} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ is a chain of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$. Besides, $\mathfrak{R}(\mathcal{L}_x) = \{x\} \ni x$. Thus, any elementary state $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ is contained in the range of some abstract trajectory $\mathcal{L}_x \in \mathbb{Ll}(\mathcal{B})$. Therefore, $\bigcup_{\mathcal{L} \in \mathbb{Ll}(\mathcal{B})} \mathfrak{R}(\mathcal{L}) = \mathfrak{B}\mathfrak{s}(\mathcal{B})$. Thus, taking into account the statement 1) of this assertion we conclude, that $\mathbb{Ll}(\mathcal{B})$ is the system of abstract trajectories from $\mathbf{Tm}(\mathcal{B})$ to $\mathfrak{B}\mathfrak{s}(\mathcal{B})$.

3) Let $\mathcal{L} \in \mathbb{Ll}(\mathcal{B})$ be a fate line of \mathcal{B} (ie \mathcal{L} is a maximum chain of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$). Then, there not exist any chain (abstract trajectory) $\mathcal{L}_1 \in \mathbb{Ll}(\mathcal{B})$ such, that $\mathcal{L} \subset \mathcal{L}_1$. Hence, \mathcal{L} is a maximum trajectory (relatively the system of abstract trajectories $\mathbb{Ll}(\mathcal{B})$).

4) Now, we are going to prove, that $\bigcup_{\mathcal{L} \in \mathbb{Ld}(\mathcal{B})} \mathfrak{R}(\mathcal{L}) = \mathfrak{B}\mathfrak{s}(\mathcal{B})$. Since $\bigcup_{\mathcal{L} \in \mathbb{Ld}(\mathcal{B})} \mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{B}\mathfrak{s}(\mathcal{B})$, we have $\bigcup_{\mathcal{L} \in \mathbb{Ld}(\mathcal{B})} \mathfrak{R}(\mathcal{L}) \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$. Thus, it remains to prove the inverse inclusion. Chose any elementary state $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$. By assertion 8.4 (item 1), the elementary state x must have an eigen fate line $\mathcal{L}_x \in \mathbb{Ld}(\mathcal{B})$. This (by definition 8.1) means, that there exist an elementary-time state $\omega_x = (t, x) \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\omega_x \in \mathcal{L}_x$. Since $(t, x) \in \mathcal{L}_x$, then $\mathcal{L}_x(t) = x$. Therefore, $x \in \mathfrak{R}(\mathcal{L}_x) \subseteq \bigcup_{\mathcal{L} \in \mathbb{Ld}(\mathcal{B})} \mathfrak{R}(\mathcal{L})$. Thus, $\bigcup_{\mathcal{L} \in \mathbb{Ld}(\mathcal{B})} \mathfrak{R}(\mathcal{L}) = \mathfrak{B}\mathfrak{s}(\mathcal{B})$. Hence, $\mathbb{Ld}(\mathcal{B})$ is a system of abstract trajectories from $\mathbf{Tm}(\mathcal{B})$ to $\mathfrak{B}\mathfrak{s}(\mathcal{B})$. Since (by item 3 of this assertion) any fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B}) \subseteq \mathbb{Ll}(\mathcal{B})$ is a maximum trajectory relatively the system of abstract trajectories $\mathbb{Ll}(\mathcal{B})$, it is the maximum trajectory relatively the narrower system of abstract trajectories $\mathbb{Ld}(\mathcal{B})$. \square

The next theorem shows, that any basic changeable set can be generated by some system of maximum trajectories.

Theorem 8.1. *For any basic changeable set \mathcal{B} the following equality is true:*

$$\mathcal{At}(\mathbf{Tm}(\mathcal{B}), \mathbb{Ld}(\mathcal{B})) = \mathcal{B}.$$

Proof. Denote: $\mathcal{R} := \mathbb{Ld}(\mathcal{B})$. We need to prove, that $\mathcal{At}(\mathcal{R}) = \mathcal{B}$.

1) By the assertion 8.5, $\mathcal{R} = \mathbb{Ld}(\mathcal{B})$ is the system of abstract trajectories from $\mathbf{Tm}(\mathcal{B}) = (\mathbf{Tm}(\mathcal{B}), \leq_{\mathcal{B}})$ to $\mathfrak{B}\mathfrak{s}(\mathcal{B})$. Hence, by the first item of the theorem 7.1,

$$\mathbf{Tm}(\mathcal{At}(\mathcal{R})) = \mathbf{Tm}(\mathcal{B}), \quad \leq_{\mathcal{At}(\mathcal{R})} = \leq_{\mathcal{B}}.$$

2) By the second item of the theorem 7.1:

$$\mathbb{B}\mathfrak{s}(\mathcal{At}(\mathcal{R})) = \bigcup_{r \in \mathcal{R}} r = \bigcup_{\mathcal{L} \in \mathbb{Ld}(\mathcal{B})} \mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}). \quad (14)$$

On the other hand, by the assertion 8.3, for any $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ the fate line $\mathcal{L}_{\omega} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ exists such, that $\omega \in \mathcal{L}_{\omega}$. Therefore, $\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \bigcup_{\mathcal{L} \in \mathbb{Ld}(\mathcal{B})} \mathcal{L} = \mathbb{B}\mathfrak{s}(\mathcal{At}(\mathcal{R}))$. And, taking into account (14) we obtain:

$$\mathbb{B}\mathfrak{s}(\mathcal{At}(\mathcal{R})) = \mathbb{B}\mathfrak{s}(\mathcal{B}).$$

3) Let us consider any elementary-time states $\omega_1 = (t_1, x_1)$, $\omega_2 = (t_2, x_2) \in \mathbb{B}\mathfrak{s}(\mathcal{B}) = \mathbb{B}\mathfrak{s}(\mathcal{At}(\mathcal{R}))$.

3.a) Suppose, that $\omega_2 \xleftarrow{\mathcal{B}} \omega_1$. By the property 7.1(4) (see properties 7.1), $\mathbf{tm}(\omega_1) \leq \mathbf{tm}(\omega_2)$. Moreover, by the assertion 8.3 (item 2) the fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$ exists such, that $\omega_1, \omega_2 \in \mathcal{L}$. Thus, by the theorem 7.1 (item 3), $\omega_2 \xleftarrow{\mathcal{At}(\mathbb{Ld}(\mathcal{B}))} \omega_1$, that is $\omega_2 \xleftarrow{\mathcal{At}(\mathcal{R})} \omega_1$.

3.b) Conversely, suppose, that $\omega_2 \xleftarrow[\mathcal{At}(\mathcal{R})]{\mathcal{B}} \omega_1$, scilicet $\omega_2 \xleftarrow[\mathcal{At}(\mathbb{L}d(\mathcal{B}))]{\mathcal{B}} \omega_1$. Then, by the theorem 7.1 (item 3), $\mathbf{tm}(\omega_1) \leq \mathbf{tm}(\omega_2)$ and there exists the fate line $\mathfrak{L} \in \mathbb{L}d(\mathcal{B})$ exists such, that $\omega_1, \omega_2 \in \mathfrak{L}$. Since the fate line \mathfrak{L} is a chain, at least one from the correlations $\omega_2 \xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_1$ or $\omega_1 \xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_2$ must be true. We shall prove, that $\omega_2 \xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_1$. Assume the contrary ($\omega_2 \not\xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_1$). Then, we have $\omega_1 \xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_2$ and $\omega_2 \neq \omega_1$ (because in the case $\omega_1 = \omega_2$ we have $\omega_2 \xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_1$). Hence, by the property 7.1(4) $\omega_1 \xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_2$. Since $\omega_1 \xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_2$ and $\omega_2 \neq \omega_1$, by the definition 7.2 we obtain, $\mathbf{tm}(\omega_2) < \mathbf{tm}(\omega_1)$. The last inequality is impossible, because we have proved, that $\mathbf{tm}(\omega_1) \leq \mathbf{tm}(\omega_2)$. Therefore, $\omega_2 \xleftarrow[\mathcal{B}]{\mathcal{B}} \omega_1$.

From the items 3.a) and 3.b) it follows, that $\xleftarrow[\mathcal{B}]{\mathcal{B}} = \xleftarrow[\mathcal{At}(\mathcal{R})]{\mathcal{B}}$ (for the bases of elementary processes on $\mathbb{B}\mathfrak{s}(\mathcal{At}(\mathcal{R})) = \mathbb{B}\mathfrak{s}(\mathcal{B})$).

Thus, the basic changeable set \mathcal{B} satisfies the conditions 1)-3) of the theorem 7.1 (for the system of abstract trajectories $\mathcal{R} = \mathbb{L}d(\mathcal{B})$), and, by this theorem, $\mathcal{At}(\mathcal{R}) = \mathcal{B}$. \square

9 Multi-figurativeness and Unification of Perception. General Definition of Changeable Set

9.1 Changeable Systems and Processes

Definition 9.1. Let \mathcal{B} be a basic changeable set. Any subset $S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ we will name a **changeable system** of the basic changeable set \mathcal{B} .

In the mechanics the elementary states can be interpreted as the states or positions of material point in various moments of time. That is why, the concept of changeable system may be considered as the abstract generalization of the notion of physical body, which, in the general case, has not constant composition.

Definition 9.2. Let \mathcal{B} be a basic changeable set. Any mapping $s : \mathbf{Tm}(\mathcal{B}) \rightarrow 2^{\mathbb{B}\mathfrak{s}(\mathcal{B})}$ such, that $s(t) \subseteq \psi(t)$, $t \in \mathbf{Tm}(\mathcal{B})$ will be referred to as a **process** of the basic changeable set \mathcal{B} .

Since primitive changeable sets can be interpreted as basic changeable set with the base of elementary processes $\xleftarrow[\mathcal{B}]{\mathcal{B}}$, the chronometric processes, introduced in the definition 5.9 can be considered as the particular cases of the processes, introduced in the definition 9.2.

Let $S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ be an arbitrary changeable system of any basic changeable set \mathcal{B} . Denote:

$$S^\sim(t) := \{x \in \mathbb{B}\mathfrak{s}(\mathcal{B}) \mid (t, x) \in S\}, \quad t \in \mathbf{Tm}(\mathcal{B}). \quad (15)$$

It is easy to see, that $S^\sim(t) \subseteq \psi(t)$, $t \in \mathbf{Tm}(\mathcal{B})$. Thus, by definition 9.2, S^\sim is a process on the basic changeable set \mathcal{B} .

Definition 9.3. The process S^\sim will be named the **process of transformations** of the changeable system S .

Assertion 9.1. Let \mathcal{B} be a basic changeable set.

1. For any changeable systems $S_1, S_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ the equality $S_1^\sim = S_2^\sim$ holds if and only if $S_1 = S_2$.
2. For an arbitrary process s of the basic changeable set \mathcal{B} a unique changeable system $S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ exists such, that $s = S^\sim$.

Proof. 1. To prove the first statement, it is enough to verify that for any $S_1, S_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ the equality $S_1^\sim = S_2^\sim$ implies the equality $S_1 = S_2$. Hence, suppose, that $S_1^\sim = S_2^\sim$. Then for any $t \in \mathbf{Tm}(\mathcal{B})$ we have $S_1^\sim(t) = S_2^\sim(t)$. Therefore, by (15), for an arbitrary $t \in \mathbf{Tm}(\mathcal{B})$ the condition $(t, x) \in S_1$ is equivalent to the condition $(t, x) \in S_2$. But, this means, that $S_1 = S_2$.

2. Let s be a process of a basic changeable set \mathcal{B} . Denote:

$$S := \{(t, x) \mid t \in \mathbf{Tm}(\mathcal{B}), x \in s(t)\} = \bigcup_{t \in \mathbf{Tm}(\mathcal{B})} (\{t\} \times s(t)),$$

where the symbol \times denotes Cartesian product of sets. Since for any pair $(t, x) \in S$ it is true $x \in s(t) \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$, we have $S \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$. Therefore, S is a changeable system of \mathcal{B} . Moreover, for any $t \in \mathbf{Tm}(\mathcal{B})$ we obtain:

$$S^\sim(t) = \{x \in \mathfrak{B}\mathfrak{s}(\mathcal{B}) \mid (t, x) \in S\} = \{x \in \mathfrak{B}\mathfrak{s}(\mathcal{B}) \mid x \in s(t)\} = s(t).$$

Consequently, $S^\sim = s$. Suppose, an other changeable system S_1 exists such, that $S_1^\sim = s$. Then, $S^\sim = S_1^\sim$, and, by the statement 1, $S = S_1$. Thus, changeable system S , satisfying $S^\sim = s$ is unique. \square

Therefore, the mapping $(\cdot)^\sim$ provides one-to-one correspondence between changeable systems and processes of any basic changeable set. Taking into account this fact, further we will “identify” changeable systems and processes of any basic changeable set, and for denotation of processes of a basic changeable set we will use letters with tilde, keeping in mind, that any process is the process of transformations of some changeable system.

We say, that a changeable system $U \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$ in a basic changeable set \mathcal{B} is a **subsystem** of a changeable system $S \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$ if and only if $U \subseteq S$. The following assertion is true:

Assertion 9.2. *Changeable system $U \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$ is a subsystem of a changeable system $S \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$ if and only if:*

$$\forall t \in \mathbf{Tm}(\mathcal{B}) \quad U^\sim(t) \subseteq S^\sim(t).$$

Proof. 1. Let $S, U \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$ and $U \subseteq S$. Then, by (15), for any $t \in \mathbf{Tm}(\mathcal{B})$ we obtain:

$$U^\sim(t) = \{x \in \mathfrak{B}\mathfrak{s}(\mathcal{B}) \mid (t, x) \in U\} \subseteq \{x \in \mathfrak{B}\mathfrak{s}(\mathcal{B}) \mid (t, x) \in S\} = S^\sim(t).$$

2. Conversely, suppose, that $U^\sim(t) \subseteq S^\sim(t)$ for any $t \in \mathbf{Tm}(\mathcal{B})$. Denote:

$$S_1 := \bigcup_{t \in \mathbf{Tm}(\mathcal{B})} \{t\} \times S^\sim(t); \quad U_1(t) := \bigcup_{t \in \mathbf{Tm}(\mathcal{B})} \{t\} \times U^\sim(t).$$

As it had been shown in the proof of statement 2 of the assertion 9.1, $S_1^\sim = S^\sim$, $U_1^\sim = U^\sim$. Therefore, by the first item of the assertion 9.1, $S_1 = S$, $U_1 = U$. Thus:

$$U = U_1 = \bigcup_{t \in \mathbf{Tm}(\mathcal{B})} \{t\} \times U^\sim(t) \subseteq \bigcup_{t \in \mathbf{Tm}(\mathcal{B})} \{t\} \times S^\sim(t) = S_1 = S.$$

\square

Definition 9.4. *We say, that the elementary state $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ of a basic changeable set \mathcal{B} belongs to a changeable system $S \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{B})$ in a time point $t \in \mathbf{Tm}(\mathcal{B})$ if and only if $x \in S^\sim(t)$.*

The fact, that elementary state of a basic changeable set \mathcal{B} belongs to a changeable system S in a time point t , will be denoted by:

$$x \in [t, \mathcal{B}] S,$$

and in the case, when the basic changeable set is clear, we will use the denotation:

$$x \in [t] S.$$

By the assertion 9.2, for any changeable systems $U, S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ the correlation $U \subseteq S$ holds if and only if for any $t \in \mathbf{Tm}(\mathcal{B})$ and $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ the condition $x \in [t] U$ assures $x \in [t] S$.

The last remark indicates that a changeable system of any basic changeable set can be interpreted as analog of the subset notion in the classic set theory, and the relation $\in [\cdot]$ can be interpreted as analog of the belonging relation of the classic set theory. However, the elementary-time state is not the complete analogue of the notion of element in the classic set theory, because knowing all the elementary-time states of a basic changeable set, we can not fully recover this basic changeable set.

It is evident, that any fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$ of a basic changeable set \mathcal{B} is the changeable system of \mathcal{B} .

Definition 9.5. *The process \mathcal{L}^\sim , generated by a fate line $\mathcal{L} \in \mathbb{Ld}(\mathcal{B})$ of a basic changeable set \mathcal{B} we name the **elementary process** of \mathcal{B} .*

The concept of elementary process can be considered as the complete analogue of the notion of element in the classic set theory, because knowing all the elementary process of a basic changeable set, we can fully recover this basic changeable set, using the theorem 8.1.

9.2 General Definition of Changeable Set

Basic changeable set can be treated as mathematical abstraction of physical processes models (in macro level) in the case, when the observations are conducted from one, fixed point (one, fixed frame of reference). But, real, physical nature is multi-figurative, because in physics (in particular in special relative theory) “picture of the world” can significantly vary, according to the frame of reference. Therefore, we obtain not one but many basic changeable sets (connected with everyone frame of reference, of the physical model under consideration). Any of these basic changeable sets can be interpreted as individual image (or area of perception) of the physical reality. Also it can be naturally assumed, that there is a natural unification between any two areas of perception, this means, that it must be defined some rule, which specifies how the object or process from one area of perception will be looked out in other area. More precisely, we equate, using certain rules, some object or process from one area of perception with the other object or process from other area of perception, saying that it is the same object, but visible from another area of perception. In the classical mechanics such “unification of perception” is defined by the Galilean group of transformations, and in the special relative theory this unification is determined by the group of the Lorentz-Poincare. It should be noted that in the both cases the unification of perception is made not at the level of objects and processes, but at the level of elementary-time states (points 4-dimensional space-time). This means that in the both cases there is assumed, that any elementary-time state, “visible” from some area of perception is “visible” from other areas. On author opinion, this assumption is too strong to construct our abstract theory. That is why, in the definition below the unification of perception is made on the level of objects and processes. We recall, that in the previous subsection it had been introduced the concept of changeable system (subset of the set $\mathbb{B}\mathfrak{s}(\mathcal{B})$), generated by basic changeable set \mathcal{B} as an abstract analog of the notion of physical object or process.

Definition 9.6. *Let $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ be an indexed family of basic changeable sets (where \mathcal{A} is the some set of indexes). The system of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ of kind:*

$$\mathfrak{U}_{\beta\alpha} : 2^{\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)} \longmapsto 2^{\mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)} \quad (\alpha, \beta \in \mathcal{A})$$

*is referred to as **unification of perception** on $\overleftarrow{\mathcal{B}}$ if and only if the following conditions are satisfied:*

1. $\mathfrak{U}_{\alpha\alpha}A \equiv A$ for any $\alpha \in \mathcal{A}$ and $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$.
(Here and further we denote by $\mathfrak{U}_{\beta\alpha}A$ the action of the mapping $\mathfrak{U}_{\beta\alpha}$ to the set $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$, that is $\mathfrak{U}_{\beta\alpha}A := \mathfrak{U}_{\beta\alpha}(A)$.)
2. Any mapping $\mathfrak{U}_{\beta\alpha}$ is a monotonous mapping of sets, ie for any $\alpha, \beta \in \mathcal{A}$ and $A, B \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$ the condition $A \subseteq B$ assures $\mathfrak{U}_{\beta\alpha}A \subseteq \mathfrak{U}_{\beta\alpha}B$.
3. For any $\alpha, \beta, \gamma \in \mathcal{A}$ and $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$ the following inclusion holds:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A \subseteq \mathfrak{U}_{\gamma\alpha}A. \quad (16)$$

In this case the mappings $\mathfrak{U}_{\alpha\beta}$ ($\alpha, \beta \in \mathcal{A}$) we name **unification mappings**, and the triple of kind:

$$\mathcal{Z} = (\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}})$$

will be named **changeable set**.

The first condition of the definition 9.6 is quite obvious. The second condition is dictated by the natural desire “to see” a subsystem of a given changeable system in a given area of perception as the subsystem of “the same” changeable system in other area of perception. In the case of classical mechanics or special relativity theory the third condition of the definition 9.6 may be transformed to the following (stronger) condition:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\alpha}A \quad (\alpha, \beta, \gamma \in \mathcal{A}, A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)) \quad (17)$$

The replacement of the equal sign by the sign inclusion is caused by the permission to “distort the picture of reality” during “transition” to other area of perception in the case of the our abstract theory. We suppose, that during this “transition” some elementary-time states may turn out to be “invisible” in other area of perception. Further this idea will be explained more detailed (see the section 11, in particular, theorem 11.1).

9.3 Remarks on the Terminology and Denotations

Let $\mathcal{Z} = (\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}})$ be a changeable set, where $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ is an indexed family of basic changeable sets and $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ is an unification of perception on $\overleftarrow{\mathcal{B}}$. Later we will use the following terms and notations:

- 1) The set \mathcal{A} will be named the **index set** of the changeable set \mathcal{Z} , and it will be denoted by $\mathcal{I}nd(\mathcal{Z})$.
- 2) For any index $\alpha \in \mathcal{I}nd(\mathcal{Z})$ the pair $(\alpha, \mathcal{B}_\alpha)$ will be referred to as **area of perception** or **frame of reference** or **lik** of the changeable set \mathcal{Z} .
- 3) The set of all areas of perception \mathcal{Z} will be denoted by $\mathcal{L}k(\mathcal{Z})$:

$$\mathcal{L}k(\mathcal{Z}) := \{(\alpha, \mathcal{B}_\alpha) \mid \alpha \in \mathcal{I}nd(\mathcal{Z})\}.$$

Areas of perception will typically be denoted by small Latin letters (l, m, k, p and so on).

- 4) For $l = (\alpha, \mathcal{B}_\alpha) \in \mathcal{L}k(\mathcal{Z})$ we introduce the following denotations:

$$\mathbf{ind}(l) := \alpha; \quad l^\wedge := \mathcal{B}_\alpha.$$

Thus, for any area of perception $l \in \mathcal{L}k(\mathcal{Z})$ the object l^\wedge is a basic changeable set.

Further, when it does not cause confusion, for any area of perception $l \in \mathcal{L}k(\mathcal{Z})$ in denotations:

$$\begin{aligned} \mathfrak{B}\mathfrak{s}(l^\wedge), \mathbb{B}\mathfrak{s}(l^\wedge), \mathbf{Tm}(l^\wedge), \leq_{l^\wedge}, <_{l^\wedge}, \\ \geq_{l^\wedge}, >_{l^\wedge}, \psi_{l^\wedge}, \leftarrow_{l^\wedge}, \mathbb{L}d(l^\wedge) \end{aligned} \quad (18)$$

the symbol “ $^\wedge$ ” will be omitted, and the following denotations will be used instead:

$$\begin{aligned} \mathfrak{B}\mathfrak{s}(l), \mathbb{B}\mathfrak{s}(l), \mathbf{Tm}(l), \leq_l, <_l, \\ \geq_l, >_l, \psi_l, \leftarrow_l, \mathbb{L}d(l). \end{aligned} \quad (19)$$

5) For any areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$ the mapping $\mathfrak{U}_{\text{ind}(m), \text{ind}(l)}$ will be denoted by $\langle m \leftarrow l, \mathcal{Z} \rangle$ or by $\langle l \rightarrow m, \mathcal{Z} \rangle$. Hence:

$$\langle m \leftarrow l, \mathcal{Z} \rangle = \langle l \rightarrow m, \mathcal{Z} \rangle = \mathfrak{U}_{\text{ind}(m), \text{ind}(l)}.$$

In the case, when the basic changeable \mathcal{Z} set is known, the symbol \mathcal{Z} in the above notations will be omitted, and the denotations “ $\langle m \leftarrow l \rangle, \langle l \rightarrow m \rangle$ ” will be used instead. Moreover, in the case, when it does not cause confusion in the notations “ $\leq_l, <_l, \geq_l, >_l, \leftarrow_l$ ” the symbol “ l ” will be omitted, and the denotations “ $\leq, <, \geq, >, \leftarrow$ ” will be used instead.

9.4 Elementary Properties of Changeable Sets

Using the definition 9.6 and notations, introduced in the subsection 9.3, we can write the following **basic properties of changeable sets**.

Properties 9.1. *In the properties 1-6 \mathcal{Z} is any changeable set and $l, m, p \in \mathcal{L}k(\mathcal{Z})$ are any areas of perception of \mathcal{Z} .*

1. $l = (\text{ind}(l), l^\wedge)$;
2. $l^\wedge = \left(\left(\left(\mathfrak{B}\mathfrak{s}(l), \leftarrow_l \right), (\mathbf{Tm}(l), \leq_l), \psi_l \right), \leftarrow_l \right)$ is a basic changeable set.
3. $\langle l \leftarrow l \rangle A = \langle l \rightarrow l \rangle A = A, A \subseteq \mathbb{B}\mathfrak{s}(l)$;
4. $\langle m \leftarrow l \rangle A = \langle l \rightarrow m \rangle A, A \subseteq \mathbb{B}\mathfrak{s}(l)$;
5. If $A \subseteq B \subseteq \mathbb{B}\mathfrak{s}(l)$, then $\langle m \leftarrow l \rangle A \subseteq \langle m \leftarrow l \rangle B$ (or, in other words, $\langle l \rightarrow m \rangle A \subseteq \langle l \rightarrow m \rangle B$);
6. $\langle p \leftarrow m \rangle \langle m \leftarrow l \rangle A \subseteq \langle p \leftarrow l \rangle A$ (or, in other words $\langle m \rightarrow p \rangle \langle l \rightarrow m \rangle A \subseteq \langle l \rightarrow p \rangle A$), where $A \subseteq \mathbb{B}\mathfrak{s}(l)$.

In the future we will not use the definition 9.6, and usually will apply the properties 9.1. The following assertions are elementary corollaries of the properties 9.1. In these assertions the symbol \mathcal{Z} denotes any changeable set.

Assertion 9.3. *For any $l, m \in \mathcal{L}k(\mathcal{Z})$ the following equality is true:*

$$\langle m \leftarrow l \rangle \emptyset = \emptyset.$$

Proof. Denote $B := \langle m \leftarrow l \rangle \emptyset \subseteq \mathbb{B}\mathfrak{s}(m)$. By properties 9.1 (6 and 3) we obtain:

$$\langle l \leftarrow m \rangle B = \langle l \leftarrow m \rangle \langle m \leftarrow l \rangle \emptyset \subseteq \langle l \leftarrow l \rangle \emptyset = \emptyset.$$

Therefore, $\langle l \leftarrow m \rangle B = \emptyset$. Since $\emptyset \subseteq B$, then, by property 9.1(5),

$$\langle l \leftarrow m \rangle \emptyset \subseteq \langle l \leftarrow m \rangle B = \emptyset,$$

that is $\langle l \leftarrow m \rangle \emptyset = \emptyset$. Hence, by properties 9.1 (3 and 6), we obtain:

$$\emptyset = \langle m \leftarrow m \rangle \emptyset \supseteq \langle m \leftarrow l \rangle \langle l \leftarrow m \rangle \emptyset = \langle m \leftarrow l \rangle \emptyset = B.$$

□

Assertion 9.4. For any $l, m \in \mathcal{Lk}(\mathcal{Z})$ and any family of changeable systems $(A_\alpha | \alpha \in \mathcal{A})$ ($A_\alpha \subseteq \mathbb{B}\mathfrak{s}(l)$, $\alpha \in \mathcal{A}$) the following inclusions take place:

- 1) $\langle m \leftarrow l \rangle \left(\bigcap_{\alpha \in \mathcal{A}} A_\alpha \right) \subseteq \bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$;
- 2) $\bigcap_{\alpha \in \mathcal{A}} A_\alpha \supseteq \langle l \leftarrow m \rangle \left(\bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha \right)$;
- 3) $\langle m \leftarrow l \rangle \left(\bigcup_{\alpha \in \mathcal{A}} A_\alpha \right) \supseteq \bigcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$.

Note, that the set of indexes \mathcal{A} in the last assertion is an arbitrary, and, in general, it does not coincide with the set of indexes in the definition 9.6.

Proof. 1) Denote $A := \bigcap_{\alpha \in \mathcal{A}} A_\alpha$. Taking into account, that $A \subseteq A_\alpha$, $\alpha \in \mathcal{A}$ and using the property 9.1(5) we obtain:

$$\langle m \leftarrow l \rangle A \subseteq \langle m \leftarrow l \rangle A_\alpha, \quad \alpha \in \mathcal{A}.$$

Thus, $\langle m \leftarrow l \rangle A \subseteq \bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$.

2) Denote: $Q := \bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$. Then $Q \subseteq \langle m \leftarrow l \rangle A_\alpha$, $\alpha \in \mathcal{A}$. Hence, by properties 9.1(6 and 3) we obtain:

$$\langle l \leftarrow m \rangle Q \subseteq \langle l \leftarrow m \rangle \langle m \leftarrow l \rangle A_\alpha \subseteq \langle l \leftarrow l \rangle A_\alpha = A_\alpha, \quad \alpha \in \mathcal{A}.$$

Hence, $\langle l \leftarrow m \rangle Q \subseteq \bigcap_{\alpha \in \mathcal{A}} A_\alpha$, and that it was necessary to prove.

3) Denote: $A := \bigcup_{\alpha \in \mathcal{A}} A_\alpha$. Taking into account, that $A_\alpha \subseteq A$, $\alpha \in \mathcal{A}$ and using the property 9.1(5) we obtain

$$\langle m \leftarrow l \rangle A_\alpha \subseteq \langle m \leftarrow l \rangle A, \quad \alpha \in \mathcal{A}.$$

Hence, $\bigcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha \subseteq \langle m \leftarrow l \rangle A$. □

10 Examples of Changeable Sets

Example 10.1. Let $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha | \alpha \in \mathcal{A})$ be an non-empty ($\mathcal{A} \neq \emptyset$) indexed family of basic changeable sets such, that $\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$ and $\mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)$ are equipotent for any $\alpha, \beta \in \mathcal{A}$, that is $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)) = \mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_\beta))$, $\alpha, \beta \in \mathcal{A}$, where $\mathbf{card}(M)$ is the *cardinality* of the set M . Let us consider any indexed family of bijections (one-to-one correspondences) $(W_{\beta\alpha} | \alpha, \beta \in \mathcal{A})$ of kind $W_{\beta\alpha} : \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) \mapsto \mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)$, satisfying the following “pseudo-group” conditions:

$$\begin{aligned} W_{\alpha\alpha}(\omega) &= \omega, & \alpha \in \mathcal{A}, \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha); \\ W_{\gamma\beta}(W_{\beta\alpha}\omega) &= W_{\gamma\alpha}(\omega), & \alpha, \beta, \gamma \in \mathcal{A}, \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha). \end{aligned} \tag{20}$$

Remark 10.1. Such family of bijections can be easily constructed by the following way.

Since $\mathcal{A} \neq \emptyset$, we can chose any (fixed) index $\alpha_0 \in \mathcal{A}$. Also chose any family of bijections $\overleftarrow{W} = (\mathcal{W}_\alpha \mid \alpha \in \mathcal{A})$ of kind $\mathcal{W}_\alpha : \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) \mapsto \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_0})$ (such family of bijections necessarily must exist, because of $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)) = \mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_\beta))$, $\alpha, \beta \in \mathcal{A}$). Denote:

$$W_{\beta\alpha}(\omega) := \mathcal{W}_\beta^{[-1]} \mathcal{W}_\alpha(\omega), \quad \alpha, \beta \in \mathcal{A}, \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha).$$

where $\mathcal{W}_\beta^{[-1]}$ is the inverse mapping to \mathcal{W}_β . It is easy to verify, that the family of bijections $(W_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ satisfies the conditions (20).

Let us put:

$$\mathfrak{U}_{\beta\alpha}A := W_{\beta\alpha}(A) = \{W_{\beta\alpha}(\omega) \mid \omega \in A\}, \quad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha).$$

It is easy to see, that the family of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ satisfies all conditions of the definition 9.6, moreover, the third condition of this definition can be replaced by more strong condition (17). Thus the triple:

$$\mathcal{Z}\mathbf{pv}(\overleftarrow{\mathcal{B}}, \overleftarrow{W}) = (\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}})$$

is a changeable set. The changeable set $\mathcal{Z}\mathbf{pv}(\overleftarrow{\mathcal{B}}, \overleftarrow{W})$ will be named a ***precisely visible changeable set***, generated by the system of basic changeable sets $\overleftarrow{\mathcal{B}}$ and the system of mappings \overleftarrow{W} .

Note, that for any areas of perception $l, m \in \mathcal{Z}\mathbf{pv}(\overleftarrow{\mathcal{B}}, \overleftarrow{W})$ the following equalities are true:

$$l^\wedge = \mathcal{B}_{\mathbf{ind}(l)}; \quad \langle m \leftarrow l \rangle A = W_{\mathbf{ind}(m), \mathbf{ind}(l)}(A), \quad A \subseteq \mathbb{B}\mathfrak{s}(l).$$

To construct the next example we need to introduce the concept of image of basic changeable set during the mapping of the set of it's elementary-time states.

Let T, X — be any sets. For any element $\omega = (t, x) \in T \times X$ we put:

$$\mathbf{tm}(\omega) := t, \quad \mathbf{bs}(\omega) := x. \quad (21)$$

Hence, for any $\omega \in T \times X$ we have, $\omega = (\mathbf{tm}(\omega), \mathbf{bs}(\omega))$. Note, that the denotations (21) have been used before, but only for elementary-time states $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{P}) \subseteq \mathbf{Tm}(\mathcal{P}) \times \mathfrak{Bs}(\mathcal{P})$, where \mathcal{P} is a primitive or basic changeable set (see subsection 7.1).

Let \mathcal{B} be a basic changeable set. Consider any mapping $U : \mathbf{Tm}(\mathcal{B}) \times \mathfrak{Bs}(\mathcal{B}) \mapsto \mathbf{Tm}(\mathcal{B}) \times X$, where X is some set. The mapping of this type will be named as ***transforming mapping***, for the \mathcal{B} .

Theorem 10.1. *For an arbitrary transforming mapping U for a basic changeable set \mathcal{B} there exists the unique basic changeable set $U[\mathcal{B}]$ satisfying the following conditions:*

1. $\mathbb{B}\mathfrak{s}(U[\mathcal{B}]) = U(\mathbb{B}\mathfrak{s}(\mathcal{B}))$;
2. $\mathbf{Tm}(U[\mathcal{B}]) = \mathbf{Tm}(\mathcal{B})$;
3. *If $w_1, w_2 \in \mathbb{B}\mathfrak{s}(U[\mathcal{B}])$ and $\mathbf{tm}(w_1) \neq \mathbf{tm}(w_2)$, then w_1 and w_2 are united by fate in $U[\mathcal{B}]$ if and only if there exist united by fate in \mathcal{B} elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $w_1 = U(\omega_1)$, $w_2 = U(\omega_2)$.*

Proof. Proof of existence.

1. Let $U : \mathbf{Tm}(\mathcal{B}) \times \mathfrak{Bs}(\mathcal{B}) \mapsto \mathbf{Tm}(\mathcal{B}) \times X$ be a transforming mapping for the basic changeable set \mathcal{B} . Denote:

$$U[\mathfrak{Bs}(\mathcal{B})] := \mathbf{bs}(U(\mathbb{B}\mathfrak{s}(\mathcal{B}))) = \{\mathbf{bs}(w) \mid w \in U(\mathbb{B}\mathfrak{s}(\mathcal{B}))\} = \{\mathbf{bs}(U(\omega)) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\}.$$

It is evident, that $U[\mathfrak{Bs}(\mathcal{B})] \subseteq X$. Let $x_1, x_2 \in U[\mathfrak{Bs}(\mathcal{B})]$. We will suppose, that $x_2 \leq_{U[\mathcal{B}]} x_1$, if and only if one of the following conditions is satisfied:

(C1) $x_1 = x_2$.

(C2) Elements x_1, x_2 can be represented in the form, $x_1 = \mathbf{bs}(U(\omega_1))$, $x_2 = \mathbf{bs}(U(\omega_2))$, where elementary-time states $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{B})$ are united by fate in \mathcal{B} and $\mathbf{tm}(U(\omega_1)) < \mathbf{tm}(U(\omega_2))$.

By definition, $\leq_{U[\mathcal{B}]}$ is reflexive relation on $U[\mathfrak{Bs}(\mathcal{B})]$. Hence, the pair $\left(U[\mathfrak{Bs}(\mathcal{B})], \leq_{U[\mathcal{B}]} \right)$ is an oriented set.

2. For an arbitrary $t \in \mathbf{Tm}(\mathcal{B})$ we put:

$$\tilde{\psi}_{U[\mathcal{B}]}(t) := \{ \mathbf{bs}(U(\omega)) \mid \omega \in \mathbb{Bs}(\mathcal{B}), \mathbf{tm}(U(\omega)) = t \} \quad (22)$$

(in particular, $\tilde{\psi}_{U[\mathcal{B}]}(t) = \emptyset$, in the case, when there not exist any elementary-time state $\omega \in \mathbb{Bs}(\mathcal{B})$ such, that $\mathbf{tm}(U(\omega)) = t$).

We are going to prove, that the mapping $\tilde{\psi}_{U[\mathcal{B}]} : U[\mathbf{Tm}(\mathcal{B})] \mapsto 2^{U[\mathfrak{Bs}(\mathcal{B})]}$ is a time on the oriented set $\left(U[\mathfrak{Bs}(\mathcal{B})], \leq_{U[\mathcal{B}]} \right)$.

2.a) Let, $x \in U[\mathfrak{Bs}(\mathcal{B})]$. Then $x = \mathbf{bs}(U(\omega))$ for some $\omega \in \mathbb{Bs}(\mathcal{B})$. Denote, $t := \mathbf{tm}(U(\omega))$. Then, by the definition of the mapping $\tilde{\psi}_{U[\mathcal{B}]}$, we have $x \in \tilde{\psi}_{U[\mathcal{B}]}(t)$. Thus, the first condition of the time definition 3.1 holds.

2.b) Suppose, that $x_1, x_2 \in U[\mathfrak{Bs}(\mathcal{B})]$, $x_2 \leq_{U[\mathcal{B}]} x_1$ and $x_1 \neq x_2$. Then, by the definition of the relation $\leq_{U[\mathcal{B}]}$, the elements x_1, x_2 can be represented in the form $x_1 = \mathbf{bs}(U(\omega_1))$, $x_2 = \mathbf{bs}(U(\omega_2))$, where $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{B})$ and $\mathbf{tm}(U(\omega_1)) < \mathbf{tm}(U(\omega_2))$. Denote, $t_1 := \mathbf{tm}(U(\omega_1))$, $t_2 := \mathbf{tm}(U(\omega_2))$. Then $t_1 < t_2$, and, by the definition of the mapping $\tilde{\psi}_{U[\mathcal{B}]}$, we obtain, $x_1 \in \tilde{\psi}_{U[\mathcal{B}]}(t_1)$, $x_2 \in \tilde{\psi}_{U[\mathcal{B}]}(t_2)$. Hence, the second condition of the definition 3.1 also is satisfied.

Thus, the triple:

$$U_p[\mathcal{B}] = \left(\left(U[\mathfrak{Bs}(\mathcal{B})], \leq_{U[\mathcal{B}]} \right), (\mathbf{Tm}(\mathcal{B}), \leq), \tilde{\psi}_{U[\mathcal{B}]} \right)$$

is a primitive changeable set, satisfying the following conditions

$$\begin{aligned} \mathfrak{Bs}(U_p[\mathcal{B}]) &= U[\mathfrak{Bs}(\mathcal{B})], \\ \mathbf{Tm}(U_p[\mathcal{B}]) &= \mathbf{Tm}(\mathcal{B}), \leq_{U_p[\mathcal{B}]} = \leq_{\mathcal{B}}, \psi_{U_p[\mathcal{B}]} = \tilde{\psi}_{U[\mathcal{B}]} \end{aligned} \quad (23)$$

3. Moreover, using the definition 7.1, correlation (23) and the definition (22) of the time $\tilde{\psi}_{U[\mathcal{B}]}$ we obtain:

$$\begin{aligned} \mathbb{Bs}(U_p[\mathcal{B}]) &= \{ (t, x) \mid t \in \mathbf{Tm}(U_p[\mathcal{B}]), x \in \psi_{U_p[\mathcal{B}]}(t) \} = \\ &= \{ (t, x) \mid t \in \mathbf{Tm}(\mathcal{B}), x \in \tilde{\psi}_{U[\mathcal{B}]}(t) \} = \\ &= \{ (t, x) \mid t \in \mathbf{Tm}(\mathcal{B}), x = \mathbf{bs}(U(\omega)), \omega \in \mathbb{Bs}(\mathcal{B}), \mathbf{tm}(U(\omega)) = t \} = \\ &= \{ (t, x) \mid x = \mathbf{bs}(U(\omega)), t = \mathbf{tm}(U(\omega)), \omega \in \mathbb{Bs}(\mathcal{B}) \} = \\ &= \{ (\mathbf{tm}(U(\omega)), \mathbf{bs}(U(\omega))) \mid \omega \in \mathbb{Bs}(\mathcal{B}) \} = \\ &= \{ U(\omega) \mid \omega \in \mathbb{Bs}(\mathcal{B}) \} = U(\mathbb{Bs}(\mathcal{B})). \end{aligned} \quad (24)$$

4. Let $w_1, w_2 \in \mathbb{B}\mathfrak{s}(U_p[\mathcal{B}]) = U(\mathbb{B}\mathfrak{s}(\mathcal{B}))$. We will consider, that $w_2 \xleftarrow{U[\mathcal{B}]} w_1$, if and only if one of the following conditions is satisfied:

(C3) $w_1 = w_2$;

(C4) $\mathbf{tm}(w_1) < \mathbf{tm}(w_2)$, while $w_1 = U(\omega_1)$, $w_2 = U(\omega_2)$, where elementary-time states ω_1, ω_2 are united by fate in \mathcal{B} .

We aim to prove, that $\xleftarrow{U[\mathcal{B}]}$ is a base of elementary processes on the primitive changeable set $U_p[\mathcal{B}]$.

4.a) By the condition (C3) for any $w \in \mathbb{B}\mathfrak{s}(U_p[\mathcal{B}])$ we have, $w \xleftarrow{U[\mathcal{B}]} w$. Consequently, the first condition of the definition 7.4 is satisfied.

4.b) Suppose, that $w_1 = (t_1, x_1)$, $w_2 = (t_2, x_2) \in \mathbb{B}\mathfrak{s}(U_p[\mathcal{B}])$ and $w_2 \xleftarrow{U[\mathcal{B}]} w_1$. In the case $w_1 = w_2$, by definition 7.2, we obtain $w_2 \xleftarrow{U_p[\mathcal{B}]} (f) w_1$. Therefore, it remains to consider the case $w_1 \neq w_2$. Since $w_2 \xleftarrow{U[\mathcal{B}]} w_1$ and $w_1 \neq w_2$, then, by the definition of the relation $\xleftarrow{U[\mathcal{B}]}$, we have $t_1 < t_2$, and $w_1 = U(\omega_1)$, $w_2 = U(\omega_2)$, where elementary-time states ω_1, ω_2 are united by fate in \mathcal{B} . Hence, by the definition of the relation $\xleftarrow{U[\mathcal{B}]}$ on $\mathbb{B}\mathfrak{s}(U_p[\mathcal{B}]) = U[\mathbb{B}\mathfrak{s}(\mathcal{B})]$, we obtain $x_2 \xleftarrow{U[\mathcal{B}]} x_1$. Therefore, $x_2 \xleftarrow{U_p[\mathcal{B}]} x_1$ and $t_1 < t_2$. Consequently, $w_2 \xleftarrow{U_p[\mathcal{B}]} (f) w_1$. Thus, the second condition of the definition 7.4 also holds.

4.c) Let, $x_1, x_2 \in \mathbb{B}\mathfrak{s}(U_p[\mathcal{B}]) = U[\mathbb{B}\mathfrak{s}(\mathcal{B})]$ and $x_2 \xleftarrow{U_p[\mathcal{B}]} x_1$ (that is $x_2 \xleftarrow{U[\mathcal{B}]} x_1$). By definition of the relation $\xleftarrow{U[\mathcal{B}]}$, the latter can occur only when at least one of the conditions (C1), (C2) is satisfied.

Case (C1): $x_1 = x_2$. Since $x_1 \in U[\mathbb{B}\mathfrak{s}(\mathcal{B})] = \mathbf{bs}(U(\mathbb{B}\mathfrak{s}(\mathcal{B})))$, there exists elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $x_1 = \mathbf{bs}(U(\omega))$. Denote, $w = U(\omega)$. Then $x_1 = x_2 = \mathbf{bs}(w)$, where, by item 4.a), $w \xleftarrow{U[\mathcal{B}]} w$. Thus, in this case, the third condition of the definition 7.4 holds.

Case (C2): $x_1 = \mathbf{bs}(U(\omega_1))$, $x_2 = \mathbf{bs}(U(\omega_2))$, where elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ are united by fate in \mathcal{B} and $\mathbf{tm}(U(\omega_1)) < \mathbf{tm}(U(\omega_2))$. Denote:

$$w_1 := U(\omega_1), \quad w_2 := U(\omega_2).$$

Then, $x_1 = \mathbf{bs}(w_1)$, $x_2 = \mathbf{bs}(w_2)$, where, by definition of the relation $\xleftarrow{U[\mathcal{B}]}$ on $\mathbb{B}\mathfrak{s}(U_p[\mathcal{B}])$ (condition (C4)), $w_2 \xleftarrow{U[\mathcal{B}]} w_1$.

Thus, the relation $\xleftarrow{U[\mathcal{B}]}$, defined on $\mathbb{B}\mathfrak{s}(U_p[\mathcal{B}])$, is the base of elementary processes. Hence, the pair

$$U[\mathcal{B}] = \left(U_p[\mathcal{B}], \xleftarrow{U[\mathcal{B}]} \right)$$

is a basic changeable set.

5. By the equality (24), $\mathbb{B}\mathfrak{s}(U[\mathcal{B}]) = \mathbb{B}\mathfrak{s}(U_p[\mathcal{B}]) = U(\mathbb{B}\mathfrak{s}(\mathcal{B}))$. Also from (23) it follows, that $\mathbf{Tm}(U[\mathcal{B}]) = (\mathbf{Tm}(U[\mathcal{B}]), \leq_{U[\mathcal{B}]}) = (\mathbf{Tm}(\mathcal{B}), \leq_{\mathcal{B}}) = \mathbf{Tm}(\mathcal{B})$. Thus, the basic changeable set $U[\mathcal{B}]$ satisfies the first two conditions of the Theorem.

We aim to prove, that the third condition of the Theorem for the basic changeable set $U[\mathcal{B}]$ also is satisfied. Consider any elementary-time states $w_1, w_2 \in \mathbb{B}\mathfrak{s}(U[\mathcal{B}])$ such, that $\mathbf{tm}(w_1) \neq \mathbf{tm}(w_2)$.

5.a) Suppose, that w_1, w_2 are united by fate in $U[\mathcal{B}]$. Since $\mathbf{tm}(w_1) \neq \mathbf{tm}(w_2)$, we have

$w_1 \neq w_2$. Since w_1, w_2 are united by fate in $U[\mathcal{B}]$, by the assertion 8.3, at least one of the conditions $w_2 \xleftarrow{U[\mathcal{B}]} w_1$ or $w_1 \xleftarrow{U[\mathcal{B}]} w_2$ must be fulfilled. For example we consider the case $w_2 \xleftarrow{U[\mathcal{B}]} w_1$ (another case is considered similarly). By definition of the base of elementary processes on $U[\mathcal{B}]$ the last relation means, that $w_2 \xleftarrow{U[\mathcal{B}]} w_1$. Since $w_2 \xleftarrow{U[\mathcal{B}]} w_1$ and $w_1 \neq w_2$, the condition (C4) must be satisfied. Thus, there exist united by fate in \mathcal{B} elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $w_1 = U(\omega_1)$, $w_2 = U(\omega_2)$, what is needed to prove.

5.b) Now, suppose, that $w_1 = U(\omega_1)$, $w_2 = U(\omega_2)$, where the elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ are united by fate in \mathcal{B} . Since $\mathbf{tm}(w_1) \neq \mathbf{tm}(w_2)$, by the condition (C4) in the case $\mathbf{tm}(w_1) < \mathbf{tm}(w_2)$ we obtain, that $w_2 \xleftarrow{U[\mathcal{B}]} w_1$, that is $w_2 \xleftarrow{U[\mathcal{B}]} w_1$, and in the case $\mathbf{tm}(w_2) < \mathbf{tm}(w_1)$ we have $w_1 \xleftarrow{U[\mathcal{B}]} w_2$. Thus, by the assertion 8.3, w_1 and w_2 are united by fate in $U[\mathcal{B}]$.

Proof of the uniqueness.

Let $U^\sim[\mathcal{B}]$ be other basic changeable set, which satisfies the conditions 1,2,3 of this theorem. Then, from first two conditions of the theorem, it follows, that $\mathbb{B}\mathfrak{s}(U[\mathcal{B}]) = \mathbb{B}\mathfrak{s}(U^\sim[\mathcal{B}])$, $\mathbf{Tm}(U[\mathcal{B}]) = \mathbf{Tm}(U^\sim[\mathcal{B}])$, $\leq_{U[\mathcal{B}]} = \leq_{U^\sim[\mathcal{B}]}$. Next, we are going to prove, that the bases of elementary processes $\xleftarrow{U[\mathcal{B}]}$ and $\xleftarrow{U^\sim[\mathcal{B}]}$ on the set $\mathbb{B}\mathfrak{s}(U[\mathcal{B}]) = \mathbb{B}\mathfrak{s}(U^\sim[\mathcal{B}])$ are identical.

Suppose, that $w_1, w_2 \in \mathbb{B}\mathfrak{s}(U[\mathcal{B}])$ and $w_2 \xleftarrow{U[\mathcal{B}]} w_1$.

In the case $w_1 = w_2$, by the definition of base of elementary processes 7.4, we have $w_2 \xleftarrow{U^\sim[\mathcal{B}]} w_1$.

Thus, it remains to consider the case $w_1 \neq w_2$. Since $w_2 \xleftarrow{U[\mathcal{B}]} w_1$, then, by the property 7.1(4), $w_2 \xleftarrow{U[\mathcal{B}]} (f) w_1$. Since $w_2 \xleftarrow{U[\mathcal{B}]} (f) w_1$ and $w_1 \neq w_2$, by definition 7.2, $\mathbf{tm}(w_1) <_{U[\mathcal{B}]} \mathbf{tm}(w_2)$. Hence, $\mathbf{tm}(w_1) \neq \mathbf{tm}(w_2)$. By the assertion 8.3, w_1 and w_2 are united by fate in $U[\mathcal{B}]$. So, taking into account, that the basic changeable set $U[\mathcal{B}]$ satisfies the third condition of this theorem, we obtain, that there exists united by fate in \mathcal{B} elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $w_1 = U(\omega_1)$, $w_2 = U(\omega_2)$. Hence, since the basic changeable set $U^\sim[\mathcal{B}]$ also satisfies the third condition of this theorem and $\mathbf{tm}(w_1) \neq \mathbf{tm}(w_2)$, w_1 and w_2 are united by fate in $U^\sim[\mathcal{B}]$. Thus, one of the conditions $w_2 \xleftarrow{U^\sim[\mathcal{B}]} w_1$ or $w_1 \xleftarrow{U^\sim[\mathcal{B}]} w_2$ must be fulfilled. Since $\mathbf{tm}(w_1) <_{U[\mathcal{B}]} \mathbf{tm}(w_2)$ and $\leq_{U[\mathcal{B}]} = \leq_{U^\sim[\mathcal{B}]}$, we have $\mathbf{tm}(w_1) <_{U^\sim[\mathcal{B}]} \mathbf{tm}(w_2)$. Thus, the condition $w_1 \xleftarrow{U^\sim[\mathcal{B}]} w_2$ is impossible by the property 7.1(4). Consequently, $w_2 \xleftarrow{U^\sim[\mathcal{B}]} w_1$.

Thus, we have proved, that for any $w_1, w_2 \in \mathbb{B}\mathfrak{s}(U[\mathcal{B}]) = \mathbb{B}\mathfrak{s}(U^\sim[\mathcal{B}])$ condition $w_2 \xleftarrow{U[\mathcal{B}]} w_1$ involves the condition $w_2 \xleftarrow{U^\sim[\mathcal{B}]} w_1$. Similarly, it can be proved, that condition $w_2 \xleftarrow{U^\sim[\mathcal{B}]} w_1$ involves the condition $w_2 \xleftarrow{U[\mathcal{B}]} w_1$. This means, that $\xleftarrow{U[\mathcal{B}]} \upharpoonright_{\mathbb{B}\mathfrak{s}(U[\mathcal{B}])} = \xleftarrow{U^\sim[\mathcal{B}]} \upharpoonright_{\mathbb{B}\mathfrak{s}(U^\sim[\mathcal{B}])}$, where $\xleftarrow{U[\mathcal{B}]} \upharpoonright_{\mathbb{B}\mathfrak{s}(U[\mathcal{B}])}$ and $\xleftarrow{U^\sim[\mathcal{B}]} \upharpoonright_{\mathbb{B}\mathfrak{s}(U^\sim[\mathcal{B}])}$ are the bases of elementary processes on $U[\mathcal{B}]$ and $U^\sim[\mathcal{B}]$ correspondingly.

Now we have proved the equalities $\mathbb{B}\mathfrak{s}(U[\mathcal{B}]) = \mathbb{B}\mathfrak{s}(U^\sim[\mathcal{B}])$, $\mathbf{Tm}(U[\mathcal{B}]) = \mathbf{Tm}(U^\sim[\mathcal{B}])$, $\leq_{U[\mathcal{B}]} = \leq_{U^\sim[\mathcal{B}]}$ and $\xleftarrow{U[\mathcal{B}]} \upharpoonright_{\mathbb{B}\mathfrak{s}(U[\mathcal{B}])} = \xleftarrow{U^\sim[\mathcal{B}]} \upharpoonright_{\mathbb{B}\mathfrak{s}(U^\sim[\mathcal{B}])}$. From these equalities it follows, that other components of the basic changeable sets $U[\mathcal{B}]$ and $U^\sim[\mathcal{B}]$ also are identical (according to the properties 7.1(5,6)). Thus, $U[\mathcal{B}] = U^\sim[\mathcal{B}]$. \square

Definition 10.1. The basic changeable set $U[\mathcal{B}]$, which satisfies the conditions 1,2,3 of the theorem 10.1 will be named the **image of the basic changeable set \mathcal{B} during the transforming mapping $U : \mathbf{Tm}(\mathcal{B}) \times \mathfrak{B}\mathfrak{s}(\mathcal{B}) \mapsto \mathbf{Tm}(\mathcal{B}) \times X$** .

Remark 10.2. According to conditions (C3),(C4) in the proof of theorem 10.1 for any elementary-time states $w_1, w_2 \in \mathbb{B}\mathfrak{s}(U[\mathcal{B}])$ the relation $w_2 \xleftarrow{U[\mathcal{B}]} w_1$ is true if and only if $w_1 = w_2$

or $\text{tm}(w_1) < \text{tm}(w_2)$ and there exists united by fate in \mathcal{B} elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $w_1 = U(\omega_1)$, $w_2 = U(\omega_2)$.

Example 10.2. Let \mathcal{B} be a basic changeable set, and X — an arbitrary set such, that $\mathfrak{B}\mathfrak{s}(\mathcal{B}) \subseteq X$. And let \mathbb{U} be any set of bijections (one-to-one correspondences) of kind:

$$U : \mathbf{Tm}(\mathcal{B}) \times X \mapsto \mathbf{Tm}(\mathcal{B}) \times X \quad (U \in \mathbb{U})$$

Such set of bijections \mathbb{U} will be referred to as *transforming set of mappins* for the basic changeable set \mathcal{B} .

Denote:

$$\begin{aligned} \mathcal{A} &:= \mathbb{U}; \\ U_\alpha &:= \alpha, \quad \alpha \in \mathcal{A} \end{aligned}$$

Then we obtain the indexed set of mappings $\overleftarrow{\mathbb{U}} = (U_\alpha \mid \alpha \in \mathcal{A})$ such, that $U_\alpha \neq U_\beta$, for $\alpha \neq \beta$.

Any mapping U_α ($\alpha \in \mathcal{A}$) is a transforming mapping, for the basic changeable set \mathcal{B} . Thus, we obtain a family of basic changeable sets:

$$\begin{aligned} \mathcal{B}_\alpha &:= U_\alpha[\mathcal{B}], \quad \alpha \in \mathcal{A}; \\ \overleftarrow{\mathbb{U}}[\mathcal{B}] &= (U_\alpha[\mathcal{B}] \mid \alpha \in \mathcal{A}) = (U[\mathcal{B}] \mid U \in \mathbb{U}). \end{aligned}$$

By theorem 10.1:

$$\mathbb{B}\mathfrak{s}(U_\alpha[\mathcal{B}]) = U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B})), \quad \alpha \in \mathcal{A},$$

so any mapping U_α is a bijection from the set $\mathbb{B}\mathfrak{s}(\mathcal{B})$ to the set $\mathbb{B}\mathfrak{s}(U_\alpha[\mathcal{B}])$. Hence, we can consider the family of mappings:

$$\tilde{U}_{\beta\alpha} := U_\beta U_\alpha^{-1} = U_\beta(U_\alpha^{-1}), \quad \alpha, \beta \in \mathcal{A}.$$

For any $\alpha, \beta \in \mathcal{A}$ the mapping $\tilde{U}_{\beta\alpha}$ is bijection from the set $\mathbb{B}\mathfrak{s}(U_\alpha[\mathcal{B}])$ to the set $\mathbb{B}\mathfrak{s}(U_\beta[\mathcal{B}])$. We are near to prove that the family of mappings $\tilde{\mathbb{U}} = (\tilde{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ satisfies the conditions (20). Indeed:

$$\begin{aligned} \tilde{U}_{\alpha\alpha}(\omega) &:= U_\alpha U_\alpha^{-1}(\omega) = \omega; \\ \tilde{U}_{\gamma\beta} \tilde{U}_{\beta\alpha}(\omega) &= U_\gamma U_\beta^{-1} U_\beta U_\alpha^{-1}(\omega) = U_\gamma U_\alpha^{-1}(\omega) = \tilde{U}_{\gamma\alpha}(\omega) \\ &\quad (\alpha, \beta, \gamma \in \mathcal{A}, \omega \in \mathbb{B}\mathfrak{s}(U_\alpha[\mathcal{B}])). \end{aligned}$$

Thus, by results of example 10.1, we can construct the changeable set:

$$\mathcal{Z}\text{im}(\mathbb{U}, \mathcal{B}) = \mathcal{Z}\text{pv}(\overleftarrow{\mathbb{U}}[\mathcal{B}], \tilde{\mathbb{U}}).$$

The changeable set $\mathcal{Z}\text{im}(\mathbb{U}, \mathcal{B})$ will be named *multi-figurative image* of the basic changeable set \mathcal{B} relatively the transforming set of mappins \mathbb{U} .

Example 10.3. Let \mathcal{B} be a basic changeable set such, that

$$\mathfrak{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbb{R}^3, \quad \mathbf{Tm}(\mathcal{B}) = \mathbb{R}$$

(for example it may be, that $\mathcal{B} = \mathcal{A}t(\mathcal{R})$, where \mathcal{R} is a system of abstract trajectories from \mathbb{R} to \mathbb{R}^3). The Poincare group $\mathbb{U} = P(1, 3)$, defined on the 4-dimensional space-time $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3 \supseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{B}\mathfrak{s}(\mathcal{B})$ is transforming set of mappins for this basic changeable set \mathcal{B} . Hence, we obtain the changeable set $\mathcal{Z}\text{im}(P(1, 3), \mathcal{B})$, which can be applied to formalization of the cinematics of special relativity theory in the inertial frames of reference.

In the examples 10.1-10.3 the unification mappings $\langle m \leftarrow l \rangle$ ($l, m \in \mathcal{L}k(\mathcal{Z})$) are defined by means of bijections (one-to-one correspondences) between the sets of elementary-time states $\mathbb{B}\mathfrak{s}(l)$ and $\mathbb{B}\mathfrak{s}(m)$ (that is $\langle m \leftarrow l \rangle A = \bigcup_{\omega \in A} \langle m \leftarrow l \rangle \{\omega\}$ and the third condition of the definition 9.6 may be replaced by more strong condition (17)). But really the conditions of the definition 9.6 are enough general. And the next examples show, how far in this definition was made a departure from the usual for physics “pointwise” comparison between elementary-time states of different areas of perception (frames of reference).

Example 10.4. Let $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ be any indexed family of basic changeable sets. Denote:

$$\mathfrak{U}_{\beta\alpha}A := \begin{cases} A, & \alpha = \beta \\ \emptyset, & \alpha \neq \beta \end{cases}, \quad \alpha, \beta \in \mathcal{A}, A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha).$$

It is easy to verify, that for the family of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ all conditions of the definition 9.6 are satisfied. Therefore, the triple

$$\mathcal{Z}\text{nv}(\overleftarrow{\mathcal{B}}) = (\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}})$$

is a changeable set.

The changeable set $\mathcal{Z}\text{nv}(\overleftarrow{\mathcal{B}})$ will be named the *fully invisible changeable set*, generated by the system of basic changeable sets $\overleftarrow{\mathcal{B}}$.

Note, that any basic changeable set \mathcal{B} can be identified with the changeable set of kind $\mathcal{Z}\text{nv}(\overleftarrow{\mathcal{B}})$, where $\mathcal{A} = \{1\}$, $\mathcal{B}_1 = \mathcal{B}$ and $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A}) = (\mathcal{B}_1)$.

Example 10.5. Let, $\overleftarrow{\mathcal{B}} = (\mathcal{B}_1, \mathcal{B}_2)$ ($\mathcal{A} = \{1, 2\}$) be a family of two basic changeable sets. Choose any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_2)$. Denote:

$$\begin{aligned} \mathfrak{U}_{11}A &:= A, \quad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1); & \mathfrak{U}_{22}A &:= A, \quad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_2); \\ \mathfrak{U}_{21}A &:= \begin{cases} \emptyset, & A \neq \mathbb{B}\mathfrak{s}(\mathcal{B}_1) \\ \{\omega\}, & A = \mathbb{B}\mathfrak{s}(\mathcal{B}_1) \end{cases}, & A &\subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1); \\ \mathfrak{U}_{12}A &:= \begin{cases} \emptyset, & \omega \notin A \\ \mathbb{B}\mathfrak{s}(\mathcal{B}_1), & \omega \in A \end{cases}, & A &\subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_2); \end{aligned}$$

1. Since $\mathfrak{U}_{11}, \mathfrak{U}_{22}$ are identity mappings of sets, the first condition of the definition 9.6 is performed by a trivial way. For the same reason the second condition of this definition also is satisfied in the case $\alpha = \beta$.

2. Suppose, that $\alpha, \beta \in \mathcal{A} = \{1, 2\}$, $A, B \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$, $A \subseteq B$. According to the remark in the end of previous item, it is enough to consider only the case $\alpha \neq \beta$. Thus, we have the next two subcases.

2.a) $\alpha = 1, \beta = 2$. In the case $A \neq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ we obtain $\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{21}A = \emptyset \subseteq \mathfrak{U}_{\beta\alpha}B$, and in the case $A = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$, since $A \subseteq B$ we have $B = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$, and, therefore, $\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\beta\alpha}B$.

2.b) $\alpha = 2, \beta = 1$. In the case $\omega \notin A$ we obtain $\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}A = \emptyset \subseteq \mathfrak{U}_{\beta\alpha}B$. In the case $\omega \in A$ from the condition $A \subseteq B$ it follows, that $\omega \in B$, so $\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}A = \mathbb{B}\mathfrak{s}(\mathcal{B}_1) = \mathfrak{U}_{12}B = \mathfrak{U}_{\beta\alpha}B$.

3. Let $\alpha, \beta, \gamma \in \mathcal{A} = \{1, 2\}$, $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$. We consider the following cases.

3.a) $\alpha = \beta$. In this case $\mathfrak{U}_{\beta\alpha}A = A$. Consequently:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\beta}A = \mathfrak{U}_{\gamma\alpha}A.$$

3.b) $\beta = \gamma$. In this case $\mathfrak{U}_{\gamma\beta}S = S$, $S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)$. Hence:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\alpha}A.$$

3.c) $\alpha \neq \beta \neq \gamma$. Since the set \mathcal{A} is two-element, this case can be divided into the following two subcases:

3.c.1) Let $\alpha = 1$, $\beta = 2$, $\gamma = 1$. Then in the case $A \neq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ we obtain:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}\mathfrak{U}_{21}A = \mathfrak{U}_{12}\emptyset = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A,$$

and in the case $A = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ we calculate:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}\mathfrak{U}_{21}A = \mathfrak{U}_{12}\{\omega\} = \mathbb{B}\mathfrak{s}(\mathcal{B}_1) = A = \mathfrak{U}_{\gamma\alpha}A.$$

3.c.2) Let, $\alpha = 2$, $\beta = 1$, $\gamma = 2$. Then in the case $\omega \notin A$ we have:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{21}\mathfrak{U}_{12}A = \mathfrak{U}_{21}\emptyset = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A,$$

and in the case $\omega \in A$ we obtain:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{21}\mathfrak{U}_{12}A = \mathfrak{U}_{21}\mathbb{B}\mathfrak{s}(\mathcal{B}_1) = \{\omega\} \subseteq A = \mathfrak{U}_{\gamma\alpha}A.$$

Consequently, the triple:

$$\mathcal{Z}_1 = (\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathcal{U}}),$$

where $\overleftarrow{\mathcal{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ is a changeable set.

Example 10.6. Let \mathcal{A} , \mathcal{B}_1 , \mathcal{B}_2 , ω be the same as in the example 10.5. As well as in the previous example 10.5, \mathfrak{U}_{11} and \mathfrak{U}_{22} are the identical mappings of the sets. Also, we denote:

$$\begin{aligned} \mathfrak{U}_{21}A &:= \begin{cases} \emptyset, & A = \emptyset \\ \{\omega\}, & A \neq \emptyset \end{cases}, & A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1); \\ \mathfrak{U}_{12}A &:= \emptyset, & A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_2). \end{aligned}$$

1,2. Since, \mathfrak{U}_{11} and \mathfrak{U}_{22} , are the identical mappings of the sets, the first condition of the definition 9.6 is satisfied by a trivial way. The second condition of this definition also is easy to verify.

3. In the cases $\alpha = \beta = \gamma$, $\alpha \neq \beta = \gamma$, $\alpha = \beta \neq \gamma$ verification of the third condition of the definition 9.6 is the same, as in the example 10.5. Thus it remains to consider the case $\alpha \neq \beta \neq \gamma$. Like the previous example we divide this case into the following two subcases:

3.1) Let, $\alpha = 1$, $\beta = 2$, $\gamma = 1$. Then:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}\mathfrak{U}_{21}A = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A.$$

3.2) Let, $\alpha = 2$, $\beta = 1$, $\gamma = 2$. Then:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{21}\mathfrak{U}_{12}A = \mathfrak{U}_{21}\emptyset = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A.$$

Thus, the triple:

$$\mathcal{Z}_2 = (\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathcal{U}}),$$

is a changeable set.

11 Visibility in Changeable Sets

11.1 Gradations of Visibility

Definition 11.1. Let \mathcal{Z} be any changeable set, and $l, m \in \mathcal{Lk}(\mathcal{Z})$ be any areas of perception of \mathcal{Z} . We say, that a changeable system $A \subseteq \mathbb{Bs}(l)$ of the area of perception l is:

1. **visible** (partially visible) from the area of perception m , if and only if $\langle m \leftarrow l \rangle A \neq \emptyset$;
2. **normally visible** from the area of perception m , if and only if an arbitrary nonempty subsystem $B \subseteq A$ of the changeable system A is visible from m (that is $\forall B : \emptyset \neq B \subseteq A \langle m \leftarrow l \rangle B \neq \emptyset$);
3. **precisely visible** from m , if and only if:
 - (a) A is normally visible from m ;
 - (b) for any family $\{A_\alpha \mid \alpha \in \mathcal{A}\} \subseteq 2^A$ of changeable subsystems A such, that $\bigsqcup_{\alpha \in \mathcal{A}} A_\alpha = A$ the following equality holds

$$\langle m \leftarrow l \rangle A = \bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha,$$

where $\bigsqcup_{\alpha \in \mathcal{A}} A_\alpha$ denotes the disjoint union of the family of sets $\{A_\alpha \mid \alpha \in \mathcal{A}\}$, that is the union $\bigcup_{\alpha \in \mathcal{A}} A_\alpha$, with additional condition $A_\alpha \cap A_\beta = \emptyset$, $\alpha \neq \beta$.

4. **invisible** from the area of perception m , if and only if $\langle m \leftarrow l \rangle A = \emptyset$;

Remark 11.1. It is apparently, that the precise visibility of the changeable system $A \subseteq \mathbb{Bs}(l)$ ($l \in \mathcal{Lk}(\mathcal{Z})$) from the area of perception $m \in \mathcal{Lk}(\mathcal{Z})$ involves the normal visibility of A from m , and the normal visibility of any changeable system $A \subseteq \mathbb{Bs}(l)$ from m involves it's visibility (partial visibility) from m .

Assertion 11.1. For any changeable set \mathcal{Z} the following properties of visibility of changeable systems are true:

1. Empty changeable system $\emptyset \subseteq \mathbb{Bs}(l)$ always is invisible from any area of perception $m \in \mathcal{Lk}(\mathcal{Z})$.
2. Any nonempty changeable system $A \subseteq \mathbb{Bs}(l)$, $A \neq \emptyset$ always is precisely visible from its own area of perception l .
3. If a changeable system $A \subseteq \mathbb{Bs}(l)$ (where $l \in \mathcal{Lk}(\mathcal{Z})$) includes a subsystem $B \subseteq A$, which is visible from area of perception $m \in \mathcal{Lk}(\mathcal{Z})$, then the changeable system A also is visible from m .
4. If a changeable system $A \subseteq \mathbb{Bs}(l)$ is normally visible (precisely visible) from area of perception m , then any nonempty subsystem $B \subseteq A$, $B \neq \emptyset$ of changeable system A also is normally visible (precisely visible) from m .

Proof. Statements 1,2,3 of this assertion follow from the assertion 9.3 and properties 9.1 of changeable sets. Statement 4 for the case of normal visibility is trivial. Thus, it remains to prove the statement 4 for the case of precise visibility. Let a changeable system $A \subseteq \mathbb{Bs}(l)$ be precisely visible from the area of perception m . Consider any changeable system B such, that $\emptyset \neq B \subseteq A$. Since precise visibility involves the normal visibility, B is normally visible from m . Suppose, that $B = \bigsqcup_{\alpha \in \mathcal{A}} B_\alpha$. Using the equalities:

$$A = B \sqcup (A \setminus B); \quad A = \bigsqcup_{\alpha \in \mathcal{A}} B_\alpha \sqcup (A \setminus B),$$

and taking into account precise visibility of the changeable system A from m , we obtain:

$$\begin{aligned} \langle m \leftarrow l \rangle A &= \langle m \leftarrow l \rangle B \sqcup \langle m \leftarrow l \rangle (A \setminus B); \\ \langle m \leftarrow l \rangle A &= \bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle B_\alpha \sqcup \langle m \leftarrow l \rangle (A \setminus B). \end{aligned}$$

Consequently, $\langle m \leftarrow l \rangle B \sqcup \langle m \leftarrow l \rangle (A \setminus B) = \bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle B_\alpha \sqcup \langle m \leftarrow l \rangle (A \setminus B)$. Hence:

$$\langle m \leftarrow l \rangle B = \bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle B_\alpha.$$

Thus, B is precisely visible from m . □

Definition 11.2. We say, that an area of perception $l \in \mathcal{Lk}(\mathcal{Z})$ is:

1. **visible** (partially visible) from the area of perception $m \in \mathcal{Lk}(\mathcal{Z})$ (denotation is $l \succ m(\mathcal{Z})$), if and only there exists at least one visible from the m changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ (that is $\exists A \subseteq \mathbb{B}\mathfrak{s}(l) \langle m \leftarrow l \rangle A \neq \emptyset$).
2. **normally visible** from the area of perception $m \in \mathcal{Lk}(\mathcal{Z})$ (denotation is $l \succ! m(\mathcal{Z})$), if and only if any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ ($A \neq \emptyset$) is normally visible from the m .
3. **precisely visible** from m (denotation is $l \succ!! m(\mathcal{Z})$), if and only if any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ ($A \neq \emptyset$) is precisely visible from the area of perception m .
4. **invisible** from the area of perception m , if and only if any changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ is invisible from the m .

In the case, when the changeable set \mathcal{Z} is known in advance in the denotations $l \succ m(\mathcal{Z})$, $l \succ! m(\mathcal{Z})$, $l \succ!! m(\mathcal{Z})$ the sequence of symbols “ (\mathcal{Z}) ” will be omitted, and the denotations $l \succ m$, $l \succ! m$, $l \succ!! m$ will be used instead.

Remark 11.2. From the remark 11.1 it follows, that for the areas of perception $l, m \in \mathcal{Lk}(\mathcal{Z})$ the next propositions are true

- if $l \succ!! m$, then $l \succ! m$;
- if $l \succ! m$, then $l \succ m$.

Thus, precise visibility involves the normal visibility and normal visibility involves visibility (partial visibility). The example 10.5 shows, that visibility do not involve the normal visibility. Indeed, we may consider the case, when in this example $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_1)) \geq 2$. In this case for the areas of perception $l_1 = (1, \mathcal{B}_1)$, $l_2 = (2, \mathcal{B}_2)$ we have, that the changeable system $\mathbb{B}\mathfrak{s}(l_1) = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ is visible from l_2 , but it is not normally visible from l_2 , because any subset $A \subset \mathbb{B}\mathfrak{s}(l_1) = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ ($A \neq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$) is invisible from l_2 . Thus, in the case $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_1)) \geq 2$ we obtain $l_1 \succ l_2$, but **not** $l_1 \succ! l_2$.

The example 10.6 shows, that normal visibility do not involve the precise visibility. In this example any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l_1)$ ($l_1 = (1, \mathcal{B}_1)$) is normally visible from the area of perception $l_2 = (2, \mathcal{B}_2)$. But, in the case $\mathbf{card}(A) \geq 2$ the changeable system A is not precisely visible from l_2 , because in this case there exist nonempty sets $A_1, A_2 \subseteq A$ such, that $A_1 \sqcup A_2 = A$, but the images of these sets ($\langle l_2 \leftarrow l_1 \rangle A_1 = \mathfrak{U}_{21}A_1 = \{\omega\}$, $\langle l_2 \leftarrow l_1 \rangle A_2 = \mathfrak{U}_{21}A_2 = \{\omega\}$) are not disjoint. Thus, in the case $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_1)) \geq 2$ we have $l_1 \succ! l_2$, but **not** $l_1 \succ!! l_2$.

In the examples 10.1, 10.2 and 10.3 any area of perception of the changeable sets $\mathcal{Z}_{\text{pv}}(\overleftarrow{\mathcal{B}}, \overleftarrow{\mathcal{W}})$ and $\mathcal{Z}_{\text{im}}(\mathbb{U}, \mathcal{B})$ is precisely visible from another.

The next three assertions immediately follow from the definitions 11.2, 11.1 and the assertion 11.1.

Assertion 11.2. *For any changeable set \mathcal{Z} the following propositions are equivalent:*

(Vi1) *Area of perception $l \in \mathcal{Lk}(\mathcal{Z})$ is visible from area of perception $m \in \mathcal{Lk}(\mathcal{Z})$ ($l \succ m$).*

(Vi2) *The set $\mathbb{B}\mathfrak{s}(l)$ of all elementary-time states of l is visible from m .*

Assertion 11.3. *For an arbitrary changeable set \mathcal{Z} the following propositions are equivalent:*

(nVi1) *Area of perception $l \in \mathcal{Lk}(\mathcal{Z})$ is normally visible from area of perception $m \in \mathcal{Lk}(\mathcal{Z})$ ($l \succ! m$).*

(nVi2) *The set $\mathbb{B}\mathfrak{s}(l)$ of all elementary-time states of l is normally visible from m .*

(nVi3) *Any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ is visible from m ($\forall A \subseteq \mathbb{B}\mathfrak{s}(l)$ ($A \neq \emptyset \Rightarrow \langle m \leftarrow l \rangle A \neq \emptyset$)).*

Assertion 11.4. *Let \mathcal{Z} — be an arbitrary changeable set. Then:*

1. *Any area of perception $l \in \mathcal{Lk}(\mathcal{Z})$ is precisely visible from itself (that is $\forall l \in \mathcal{Lk}(\mathcal{Z})$ $l \succ!! l$).*

2. *The following propositions are equivalent:*

(pVi1) *Area of perception $l \in \mathcal{Lk}(\mathcal{Z})$ is precisely visible from area of perception $m \in \mathcal{Lk}(\mathcal{Z})$ ($l \succ!! m$).*

(pVi2) *The set $\mathbb{B}\mathfrak{s}(l)$ of all elementary-time states of l is precisely visible from m .*

Assertion 11.5. *For any changeable set \mathcal{Z} the binary relation $\succ!$ quasi order on the set $\mathcal{Lk}(\mathcal{Z})$ of all areas of perception \mathcal{Z} .*

Proof. Reflexivity of the relation $\succ!$ follows from the first item of the assertion 11.4 and from the remark 11.2. Thus, we need to prove the transitivity of the relation $\succ!$.

Suppose, that $l \succ! m$ and $m \succ! p$, where $l, m, p \in \mathcal{Lk}(\mathcal{Z})$. Then, using the assertion 11.3 (equivalence between (nVi1) and (nVi3)), for any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ we obtain, $\langle p \leftarrow l \rangle A \supseteq \langle p \leftarrow m \rangle \langle m \leftarrow l \rangle A \neq \emptyset$, thus, by the assertion 11.3, $l \succ! p$. \square

Remark 11.3. First item of the assertion 11.4 and remark 11.2 also bring about the reflexivity of the relations $\succ!!$ and \succ on the set $\mathcal{Lk}(\mathcal{Z})$ (for any changeable set \mathcal{Z}). But these relations, in general, are not transitive. And the next examples explain the last statement.

Example 11.1. Let \mathcal{B} be any basic changeable set. We consider the family $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathbb{N})$ of basic changeable sets, which is defined as follows:

$$\mathcal{B}_\alpha := \mathcal{B}, \quad \alpha \in \mathbb{N}.$$

For $\alpha, \beta \in \mathbb{N}$ we define the mappings $\mathfrak{U}_{\beta\alpha} : \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) \mapsto \mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)$ by the following way:

$$\mathfrak{U}_{\beta\alpha} A := \begin{cases} A, & \beta \in \{\alpha, \alpha + 1\}; \\ \mathbb{B}\mathfrak{s}(\mathcal{B}), & \beta > \alpha + 1, A \neq \emptyset; \\ \emptyset, & \beta > \alpha + 1, A = \emptyset; \\ \emptyset, & \beta < \alpha, \end{cases} \quad (A \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) = \mathbb{B}\mathfrak{s}(\mathcal{B}), n \in \mathbb{N}) \quad (25)$$

(where the symbols $<$, $>$ denote the usual order on the set of natural numbers).

We shell prove, that the system of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathbb{N})$ is unification of perception.

The first two conditions of the definition 9.6 for the system of mappings $\overleftarrow{\mathfrak{U}}$ are performed by a trivial way. Thus, we need to verify the third condition of this definition. Let $\alpha, \beta, \gamma \in \mathbb{N}$ and $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) = \mathbb{B}\mathfrak{s}(\mathcal{B})$. Then in the case $\alpha \leq \beta \leq \gamma$, by (25), we obtain:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \begin{cases} \emptyset, & A = \emptyset; \\ A, & A \neq \emptyset, \beta \in \{\alpha, \alpha+1\}, \gamma \in \{\beta, \beta+1\}; \\ \mathbb{B}\mathfrak{s}(\mathcal{B}), & A \neq \emptyset, \text{ and } (\beta > \alpha+1 \text{ or } \gamma > \beta+1). \end{cases} \quad (26)$$

Since $\mathfrak{U}_{\gamma\alpha}A \in \{A, \mathbb{B}\mathfrak{s}(\mathcal{B})\}$ for $\alpha \leq \gamma$, in the first two cases of the formula (26) the inclusion $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A \subseteq \mathfrak{U}_{\gamma\alpha}A$ holds. In the third case of the formula (26) we have $\gamma > \alpha+1$, and hence, $\mathfrak{U}_{\gamma\alpha}A = \mathbb{B}\mathfrak{s}(\mathcal{B})$. Thus, in this case, the last inclusion also is performed. If the condition $\alpha \leq \beta \leq \gamma$ is not satisfied, we have $\alpha > \beta$ or $\beta > \gamma$. Therefore, by the formula (25), we have, $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \emptyset$. Consequently, in this case we also have the inclusion $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A \subseteq \mathfrak{U}_{\gamma\alpha}A$. Thus, all conditions of the definition 9.6 are satisfied.

Hence, the triple $\mathcal{Z} = (\mathbb{N}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}})$ is a changeable set. For this changeable set \mathcal{Z} we have:

$$\begin{aligned} \mathcal{L}k(\mathcal{Z}) &= \{l_n \mid n \in \mathbb{N}\}, \text{ where} \\ l_n &= (n, \mathcal{B}_n) = (n, \mathcal{B}), \quad n \in \mathbb{N}, \end{aligned}$$

and for $l_n, l_m \in \mathcal{L}k(\mathcal{Z})$ the equality $\langle l_n \leftarrow l_m \rangle = \mathfrak{U}_{nm}$ holds. Thus, by (25):

$$\begin{aligned} \langle l_{n+1} \leftarrow l_n \rangle A &= A, \quad A \subseteq \mathbb{B}\mathfrak{s}(l_n) = \mathbb{B}\mathfrak{s}(\mathcal{B}), \quad n \in \mathbb{N}; \\ \langle l_{n+2} \leftarrow l_n \rangle A &= \begin{cases} \mathbb{B}\mathfrak{s}(\mathcal{B}), & A \neq \emptyset \\ \emptyset, & A = \emptyset \end{cases}, \quad A \subseteq \mathbb{B}\mathfrak{s}(l_n) = \mathbb{B}\mathfrak{s}(\mathcal{B}), \quad n \in \mathbb{N}; \end{aligned}$$

The last equality shows, that $l_n \succ!! l_{n+1}$ ($n \in \mathbb{N}$). But, in the case $\text{card}(\mathbb{B}\mathfrak{s}(\mathcal{B})) \geq 2$, l_n is normally visible, but not precisely visible from l_{n+2} . Thus, in the case $\text{card}(\mathbb{B}\mathfrak{s}(\mathcal{B})) \geq 2$ for any $n \in \mathbb{N}$ we have $l_n \succ!! l_{n+1}$, $l_{n+1} \succ!! l_{n+2}$, although the correlation $l_n \succ!! l_{n+2}$ is not true.

Example 11.2. Let basic changeable set \mathcal{B} be such, that the set $\mathbb{B}\mathfrak{s}(\mathcal{B})$ is infinite. Then there exists the sequence $(\omega_n)_{n=1}^\infty \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ of elementary-time states such, that $\omega_n \neq \omega_m$, $m \neq n$. Denote:

$$\begin{aligned} \mathcal{B}_\alpha &:= \mathcal{B}, \quad \alpha \in \mathbb{N}; \quad \overleftarrow{\mathcal{B}} := (\mathcal{B}_\alpha \mid \alpha \in \mathbb{N}); \\ \mathfrak{U}_{\beta\alpha}A &:= \begin{cases} A, & \beta = \alpha \\ \{\omega_\beta\}, & \beta = \alpha+1, \omega_\beta \in A \\ \emptyset, & \beta = \alpha+1, \omega_\beta \notin A \\ \emptyset, & \beta \notin \{\alpha, \alpha+1\}. \end{cases} \quad (A \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) = \mathbb{B}\mathfrak{s}(\mathcal{B}), \quad n \in \mathbb{N}) \end{aligned} \quad (27)$$

We shell prove, that the system of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathbb{N})$ is unification of perception. The first two conditions of the definition 9.6 for the system of mappings $\overleftarrow{\mathfrak{U}}$ are performed by a trivial way. Thus, we need to verify the third condition of this definition. Let $\alpha, \beta, \gamma \in \mathbb{N}$. It should be noted, that from (27) it follows, that $\mathfrak{U}_{\beta\alpha}\emptyset = \emptyset$ for any $\alpha, \beta \in \mathbb{N}$. Thus, according to (27), if one of the conditions $\alpha \leq \beta$ or $\beta \leq \gamma$, are not performed, then we have $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A$, $A \in \mathbb{B}\mathfrak{s}(\mathcal{B})$. Hence, we shell consider the case $\alpha \leq \beta \leq \gamma$. In the case, when $\alpha = \beta$ or $\beta = \gamma$, similarly to the example 9.6, we obtain $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\alpha}A$. Thus, it remains

to consider only the case $\alpha < \beta < \gamma$. In the cases $\beta > \alpha + 1$ or $\gamma > \beta + 1$, by (27), we obtain $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A$, $A \in \mathbb{B}\mathfrak{s}(\mathcal{B})$. Hence, it remains only the case $\beta = \alpha + 1$ and $\gamma = \beta + 1$. If $\omega_\beta \notin A$, then, by (27), $\mathfrak{U}_{\beta\alpha}A = \emptyset$, and we have, $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A$. And in the case $\omega_\beta \in A$, we obtain $\omega_\gamma = \omega_{\beta+1} \notin \{\omega_\beta\}$. Thus, in this case:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\beta}\{\omega_\beta\} = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A.$$

Consequently, the triple $\mathcal{Z} = (\mathbb{N}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathcal{U}})$ is a changeable set, satisfying:

$$\begin{aligned} \mathcal{L}k(\mathcal{Z}) &= \{l_n \mid n \in \mathbb{N}\}, \text{ where } l_n = (n, \mathcal{B}_n) = (n, \mathcal{B}), \quad n \in \mathbb{N}, \\ \langle l_n \leftarrow l_m \rangle &= \mathfrak{U}_{nm}, \quad m, n \in \mathbb{N} \quad (l_n, l_m \in \mathcal{L}k(\mathcal{Z})). \end{aligned}$$

From (27) it follows, that any $n \in \mathbb{N}$ $\langle l_{n+2} \leftarrow l_n \rangle A = \mathfrak{U}_{n+2,n}A = \emptyset$, $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}) = \mathbb{B}\mathfrak{s}(l_n)$, but, under the condition, $\omega_{n+1}, \omega_{n+2} \in A$ we have $\langle l_{n+1} \leftarrow l_n \rangle A = \{\omega_{n+1}\} \neq \emptyset$, $\langle l_{n+2} \leftarrow l_{n+1} \rangle A = \{\omega_{n+2}\} \neq \emptyset$. Therefore, $l_n \succ l_{n+1}$, $l_{n+1} \succ l_{n+2}$, although the area of perception l_n invisible from l_{n+2} ($l_n \not\succ l_{n+2}$).

Definition 11.3. We say, that a changeable set \mathcal{Z} is *visible (normally visible, precisely visible)* if and only if for any $l, m \in \mathcal{L}k(\mathcal{Z})$ it satisfied the condition $l \succ m$ ($l \succ! m$, $l \succ!! m$) correspondingly.

From remark 11.2 it follows, that any normally visible changeable set is visible. The example 10.5 shows, that the inverse assertion is not true. Indeed, we may consider the case, when in this example $\text{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_1)) \geq 2$. As it has been shown in the remark 11.2, in this case for the areas of perception $l_1 = (1, \mathcal{B}_1)$, $l_2 = (2, \mathcal{B}_2)$ we have, $l_1 \succ l_2$, but **not** $l_1 \succ! l_2$. Since in this example $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_2)$, we obtain $\langle l_1 \leftarrow l_2 \rangle \mathbb{B}\mathfrak{s}(\mathcal{B}_2) = \mathfrak{U}_{12}\mathbb{B}\mathfrak{s}(\mathcal{B}_2) = \mathbb{B}\mathfrak{s}(\mathcal{B}_1) \neq \emptyset$. Hence, $l_2 \succ l_1$. Thus $l_1 \succ l_2$, $l_2 \succ l_1$, but **not** $l_1 \succ! l_2$. And, taking into account, that $\mathcal{L}k(\mathcal{Z}_1) = \{l_1, l_2\}$, we obtain, that the changeable set \mathcal{Z}_1 in the example 10.5 is visible, but not normally visible. In the subsection 11.2 (corollary 11.1) it will be shown, that the changeable set \mathcal{Z} is precisely visible if and only if it is normally visible.

11.2 Visibility Classes

Assertion 11.6. For any areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$ of any changeable set \mathcal{Z} the following propositions are equivalent:

- (I) $l \succ! m$ and $m \succ! l$;
- (II) $l \succ!! m$ and $m \succ!! l$.

Proof. Since precise visibility always involves normal visibility, it is enough only to prove the implication (I) \Rightarrow (II). Hence, suppose, that $l, m \in \mathcal{L}k(\mathcal{Z})$, $l \succ! m$, $m \succ! l$.

1) First we shall prove, that for any $A, B \subseteq \mathbb{B}\mathfrak{s}(l)$, the equality $A \cap B = \emptyset$ is true if and only if $\langle m \leftarrow l \rangle A \cap \langle m \leftarrow l \rangle B = \emptyset$. Suppose, that $A \cap B = \emptyset$. Then, according to second item of the assertion 9.4, $\emptyset = A \cap B \supseteq \langle l \leftarrow m \rangle (\langle m \leftarrow l \rangle A \cap \langle m \leftarrow l \rangle B)$. Since $m \succ! l$ and $\langle l \leftarrow m \rangle (\langle m \leftarrow l \rangle A \cap \langle m \leftarrow l \rangle B) = \emptyset$, then, by the definition of normal visibility, $\langle m \leftarrow l \rangle A \cap \langle m \leftarrow l \rangle B = \emptyset$, what is necessary to prove. Conversely, let $\langle m \leftarrow l \rangle A \cap \langle m \leftarrow l \rangle B = \emptyset$. Then, by first item of the assertion 9.4, $\langle m \leftarrow l \rangle (A \cap B) \subseteq \langle m \leftarrow l \rangle A \cap \langle m \leftarrow l \rangle B = \emptyset$. Since $\langle m \leftarrow l \rangle (A \cap B) = \emptyset$ and $l \succ! m$, then, by the definition of normal visibility, $A \cap B = \emptyset$.

2) Let, $A \in \mathbb{B}\mathfrak{s}(l)$ and $A = \bigsqcup_{\alpha \in \mathcal{A}} A_\alpha$ (where $A_\alpha \subseteq A$, $\alpha \in \mathcal{A}$; $A_\alpha \cap A_\beta = \emptyset$, $\alpha \neq \beta$). By the item 3) of the assertion 9.4, $\langle m \leftarrow l \rangle A \supseteq \bigcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$. Since the family of sets $(A_\alpha \mid \alpha \in \mathcal{A})$ is disjoint, by the first item of this proof, the family of sets $(\langle m \leftarrow l \rangle A_\alpha \mid \alpha \in \mathcal{A})$ also is disjoint, that is $\langle m \leftarrow l \rangle A \supseteq \bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$. Assume, that the last inclusion is strict (ie $\langle m \leftarrow l \rangle A \neq \bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$). Then the set $\tilde{B} = (\langle m \leftarrow l \rangle A) \setminus (\bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha)$ is

nonempty. Hence, by the definition of normal visibility, the set $B = \langle l \leftarrow m \rangle \tilde{B}$ also is nonempty. Since $\tilde{B} \subseteq \langle m \leftarrow l \rangle A$, by the properties 9.1, $B = \langle l \leftarrow m \rangle \tilde{B} \subseteq \langle l \leftarrow m \rangle \langle m \leftarrow l \rangle A \subseteq \langle l \leftarrow l \rangle A = A$. Since the set $\tilde{B} = (\langle m \leftarrow l \rangle A) \setminus (\bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha)$ is disjoint with any of the sets $\langle m \leftarrow l \rangle A_\alpha$ ($\alpha \in \mathcal{A}$), the set $\langle m \leftarrow l \rangle B = \langle m \leftarrow l \rangle \langle l \leftarrow m \rangle \tilde{B} \subseteq \langle m \leftarrow m \rangle \tilde{B} = \tilde{B}$ also is disjoint with any of $\langle m \leftarrow l \rangle A_\alpha$ ($\alpha \in \mathcal{A}$) (ie $\langle m \leftarrow l \rangle B \cap \langle m \leftarrow l \rangle A_\alpha = \emptyset$, $\alpha \in \mathcal{A}$). Hence, by the first item of this proof, $B \cap A_\alpha = \emptyset$, $\alpha \in \mathcal{A}$. Thus, we can conclude, that there exist the **nonempty** set $B \subseteq A$ such, that $B \cap A_\alpha = \emptyset$, $\alpha \in \mathcal{A}$, which contradicts the equality $A = \bigsqcup_{\alpha \in \mathcal{A}} A_\alpha$. Thus, the assumption above is wrong, and, consequently, we obtain $\langle m \leftarrow l \rangle A = \bigsqcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$.

Thus, any set $A \subseteq \mathbb{B}\mathfrak{s}(l)$ is precisely visible from the area of perception m , ie $l \succ!! m$. Similarly, we obtain, that $m \succ!! l$. \square

The next corollary immediately follows from the assertion 11.6.

Corollary 11.1. *Changeable set \mathcal{Z} is precisely visible if and only if it is normally visible.*

Taking into account the corollary 11.1, the notion “normally visible changeable set” will be not used henceforth.

Definition 11.4. *We say, that areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$ are **equivalent respectively the precise visibility** (or, abbreviated, **precisely-equivalent**) if and only if it is satisfied the condition (II) (or, equivalently, the condition (I)) of the assertion 11.6.*

The fact, that areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$ are precisely-equivalent will be denoted by the following way:

$$l \equiv! m(\mathcal{Z}).$$

And in the case, when changeable set \mathcal{Z} known in advance we shall use the denotation $l \equiv! m$ instead.

Assertion 11.7. *Relation $\equiv!$ is relation of equivalence on the set $\mathcal{L}k(\mathcal{Z})$.*

Proof. For $l, m \in \mathcal{L}k(\mathcal{Z})$ condition $l \equiv! m$ is equivalent to the condition (I) of the assertion 11.6. Thus, since (by the assertion 11.5) the relation $\succ!$ is quasi order on $\mathcal{L}k(\mathcal{Z})$, the desired result follows from [14, page. 21]. \square

Definition 11.5. *Equivalence classes, generated by the relation $\equiv!$ will be referred to as **precise visibility classes** of the changeable set \mathcal{Z} .*

Thus, for any changeable set, the set of all its areas of perception can be splitted on the precise visibility classes. Within an arbitrary precise visibility class any area of perception is precisely visible from other. It is evident, that changeable set \mathcal{Z} is precisely visible if and only if $\mathcal{L}k(\mathcal{Z})$ contains only one precise visibility class.

It turns out, that, using the relation of visibility “ \succ ”, we can also divide the set $\mathcal{L}k(\mathcal{Z})$ by equivalence classes.

Definition 11.6. *Let \mathcal{Z} be a changeable set.*

(a) *We say, that areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$ are **directly connected by visibility** (denotation is $l \prec\succ m(\mathcal{Z})$, or $l \prec\succ m$ in the case, when changeable set \mathcal{Z} known in advance) if and only if at least one of the following conditions is satisfied:*

$$l \succ m \quad \text{or} \quad m \succ l.$$

(b) *We say, that areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$ are **connected by visibility** (denotation is $l \equiv m(\mathcal{Z})$, or $l \equiv m$ in the case, when changeable set \mathcal{Z} known in advance) if and only if there exists a sequence $l_0, l_1, \dots, l_\nu \in \mathcal{L}k(\mathcal{Z})$ ($\nu \in \mathbb{N}$) such, that:*

$$l_0 = l, \quad l_\nu = m, \quad \text{and} \quad l_i \prec\succ l_{i-1} \quad (i \in \overline{1, \nu}).$$

Assertion 11.8. *Relation $\hat{=}$ is relation of equivalence on the set $\mathcal{L}k(\mathcal{Z})$.*

Proof. Since the relation of visibility is reflexive, the relation $\prec\succ$ is reflexive and symmetric on $\mathcal{L}k(\mathcal{Z})$. The relation $\hat{=}$ is transitive closure of the relation $\prec\succ$ in the sense of [15, page 69], [16, page. 32]. Thus, by [15, assertion 5.8, 5.9; theorem 5.8], $\hat{=}$ is equivalence relation on $\mathcal{L}k(\mathcal{Z})$. \square

Definition 11.7. *Equivalence classes in the set $\mathcal{L}k(\mathcal{Z})$, generated by the relation $\hat{=}$ will be referred to as **visibility classes** of the changeable set \mathcal{Z} .*

But it may occur, that in the changeable set only one visibility class exist.

Definition 11.8. *We say, that a changeable set \mathcal{Z} is **connected visible** if and only if for any $l, m \in \mathcal{L}k(\mathcal{Z})$ it is true the correlation $l \hat{=} m$.*

It is evident, that any visible changeable set is connected visible. Analyzing the examples 11.1 and 11.2 it is easy to verify that the inverse proposition, in general, is false.

So, we see, that in the case, when a changeable set \mathcal{Z} is not connected visible the set of all it's areas of perception is splitted by "parallel worlds" (visibility classes) and any visibility class is "fully invisible" from other visibility classes. As formal example of changeable set with many visibility classes it can be considered the changeable set $\mathcal{Z}_{nv}(\overleftarrow{\mathcal{B}})$ (see example 10.4) with $\text{card}(\overleftarrow{\mathcal{B}}) \geq 2$. In the changeable set $\mathcal{Z}_{nv}(\overleftarrow{\mathcal{B}})$ any area of perception forms the separated visibility class.

Precise visibility classes also can be interpreted as "parallel worlds". But these "parallel worlds" may be partially visible from other "parallel worlds".

11.3 Precisely Visible Changeable Sets

In the classical mechanics and special relativity theory it is supposed, that any elementary-time state (or "physical event") is visible in any frame of reference. Hence, the precisely visible changeable sets are to be important for physics. In this subsection we investigate precisely visible changeable sets in more details. The changeable sets $\mathcal{Z}_{pv}(\overleftarrow{\mathcal{B}}, \overleftarrow{\mathcal{W}})$ and $\mathcal{Z}_{im}(\mathbb{U}, \mathcal{B})$, introduced in the examples 10.1, 10.2 and 10.3, evidently are precisely visible.

Remark 11.4. It should be noted, that by the assertion 11.6 and definition of the relation $\hat{=}$, for any changeable set \mathcal{Z} the following propositions are equivalent:

- (I) \mathcal{Z} is precisely visible changeable set;
- (II) for any $l, m \in \mathcal{L}k(\mathcal{Z})$ it is performed the condition $l \succ!! m$;
- (III) for any $l, m \in \mathcal{L}k(\mathcal{Z})$ it is performed the condition $l \succ! m$;
- (IV) for any $l, m \in \mathcal{L}k(\mathcal{Z})$ it is performed the condition $l \hat{=} m$.

Note also that in the first item of the proof of assertion 11.6 it was proved, the following lemma.

Lemma 11.1. *Let \mathcal{Z} be a precisely visible changeable set. Then for any $l, m \in \mathcal{L}k(\mathcal{Z})$ and $A, B \subseteq \mathbb{B}\mathfrak{s}(l)$ the equality $\langle m \leftarrow l \rangle A \cap \langle m \leftarrow l \rangle B = \emptyset$ is true if and only if $A \cap B = \emptyset$.*

Theorem 11.1. *Changeable set \mathcal{Z} is precisely visible if and only if for any $l, m, p \in \mathcal{L}k(\mathcal{Z})$ the followind equality is true:*

$$\langle p \leftarrow m \rangle \langle m \leftarrow l \rangle = \langle p \leftarrow l \rangle. \quad (28)$$

Proof. Sufficiency. Suppose, that for any $l, m, p \in \mathcal{L}k(\mathcal{Z})$ the equality (28) holds. Chose any areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$ and any changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ such, that $A \neq \emptyset$. Then, by (28),

$$A = \langle l \leftarrow l \rangle A = \langle l \leftarrow m \rangle \langle m \leftarrow l \rangle A.$$

Therefore, by the assertion 9.3, $\langle m \leftarrow l \rangle A \neq \emptyset$. Thus, by the assertion 11.3, $l \succ! m$ (for any areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$). Hence, by the remark 11.4, the changeable set \mathcal{Z} is precisely visible.

Necessity. Conversely, suppose, that the changeable set \mathcal{Z} is precisely visible. Consider any areas of perception $l, m, p \in \mathcal{L}k(\mathcal{Z})$ and any changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$. By the properties 9.1 $\langle p \leftarrow m \rangle \langle m \leftarrow l \rangle A \subseteq \langle p \leftarrow l \rangle A$. Denote:

$$B_1 := \langle p \leftarrow l \rangle A \setminus \langle p \leftarrow m \rangle \langle m \leftarrow l \rangle A.$$

Then, $B_1 \subseteq \langle p \leftarrow l \rangle A$ and $B_1 \cap \langle p \leftarrow m \rangle \langle m \leftarrow l \rangle A = \emptyset$. Denote $B := \langle l \leftarrow m \rangle \langle m \leftarrow p \rangle B_1$. Using the properties 9.1 we obtain:

$$\begin{aligned} B &= \langle l \leftarrow m \rangle \langle m \leftarrow p \rangle B_1 \subseteq \langle l \leftarrow m \rangle \langle m \leftarrow p \rangle \langle p \leftarrow l \rangle A \subseteq \\ &\subseteq \langle l \leftarrow l \rangle A = A; \\ \langle p \leftarrow m \rangle \langle m \leftarrow l \rangle B &= \langle p \leftarrow m \rangle \langle m \leftarrow l \rangle \langle l \leftarrow m \rangle \langle m \leftarrow p \rangle B_1 \subseteq \\ &\subseteq \langle p \leftarrow p \rangle B_1 = B_1. \end{aligned}$$

Hence, since $B_1 \cap \langle p \leftarrow l \rangle \langle m \leftarrow l \rangle A = \emptyset$, we have $\langle p \leftarrow m \rangle \langle m \leftarrow l \rangle B \cap \langle p \leftarrow m \rangle \langle m \leftarrow l \rangle A = \emptyset$. Consequently, using lemma 11.1, we obtain $B \cap A = \emptyset$. Since $B \subseteq A$ and $B \cap A = \emptyset$, we obtain $B = \emptyset$. Since $\langle l \leftarrow m \rangle \langle m \leftarrow p \rangle B_1 = B = \emptyset$, taking into account, that, by remark 11.4 $p \succ! m$ and $m \succ! l$, we obtain (by definition of normal visibility) $B_1 = \emptyset$. \square

Note, that, for the changeable set $\mathcal{Z} = (\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathcal{U}})$ from the definition 9.6, the condition (28) is equivalent to the condition (17).

Assertion 11.9. *Let \mathcal{Z} be a precisely visible changeable set. Then for any areas of perception $l, m \in \mathcal{L}k(\mathcal{Z})$, any family of changeable systems $(A_\alpha | \alpha \in \mathcal{A})$ ($A_\alpha \subseteq \mathbb{B}\mathfrak{s}(l)$, $\alpha \in \mathcal{A}$) and any changeable systems $A, B \in \mathbb{B}\mathfrak{s}(l)$ the following assertions are true:*

1. $\langle m \leftarrow l \rangle \left(\bigcap_{\alpha \in \mathcal{A}} A_\alpha \right) = \bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$;
2. $\langle m \leftarrow l \rangle (A \setminus B) = \langle m \leftarrow l \rangle A \setminus \langle m \leftarrow l \rangle B$;
3. $\langle m \leftarrow l \rangle \mathbb{B}\mathfrak{s}(l) = \mathbb{B}\mathfrak{s}(m)$;
4. $\langle m \leftarrow l \rangle \left(\bigcup_{\alpha \in \mathcal{A}} A_\alpha \right) = \bigcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$;
5. *If a changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ is a singleton ($\text{card}(A) = 1$), then the changeable system $\langle m \leftarrow l \rangle A$ also is a singleton.*

Proof. 1) Using the assertion 9.4, item 2), properties 9.1 and theorem 11.1 (equality (28)) we obtain:

$$\begin{aligned} \langle m \leftarrow l \rangle \left(\bigcap_{\alpha \in \mathcal{A}} A_\alpha \right) &\supseteq \langle m \leftarrow l \rangle \langle l \leftarrow m \rangle \left(\bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha \right) = \\ &= \langle m \leftarrow m \rangle \left(\bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha \right) = \bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha. \end{aligned}$$

Hence, $\langle m \leftarrow l \rangle (\bigcap_{\alpha \in \mathcal{A}} A_\alpha) \supseteq \bigcap_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha$. The inverse inclusion had been proved in the assertion 9.4, item 1).

2) Since $A \setminus B \subseteq A$, then by the property 9.1(5) we have, $\langle m \leftarrow l \rangle (A \setminus B) \subseteq \langle m \leftarrow l \rangle A$. Since $(A \setminus B) \cap B = \emptyset$, then, by lemma 11.1, $\langle m \leftarrow l \rangle (A \setminus B) \cap \langle m \leftarrow l \rangle B = \emptyset$. Hence:

$$\langle m \leftarrow l \rangle (A \setminus B) \subseteq \langle m \leftarrow l \rangle A \setminus \langle m \leftarrow l \rangle B. \quad (29)$$

Using the correlation (29) to the sets $\langle m \leftarrow l \rangle A$, $\langle m \leftarrow l \rangle B$, with unification mapping $\langle l \leftarrow m \rangle$, applying the formula (28) and properties 9.1 we obtain:

$$\begin{aligned} & \langle l \leftarrow m \rangle (\langle m \leftarrow l \rangle A \setminus \langle m \leftarrow l \rangle B) \subseteq \\ & \subseteq \langle l \leftarrow m \rangle \langle m \leftarrow l \rangle A \setminus \langle l \leftarrow m \rangle \langle m \leftarrow l \rangle B = \langle l \leftarrow l \rangle A \setminus \langle l \leftarrow l \rangle B = A \setminus B. \end{aligned}$$

Hence, by the property 9.1(5) $\langle m \leftarrow l \rangle \langle l \leftarrow m \rangle (\langle m \leftarrow l \rangle A \setminus \langle m \leftarrow l \rangle B) \subseteq \langle m \leftarrow l \rangle (A \setminus B)$. And applying the formula (28), we obtain the inverse inclusion to (29).

3) By definition of unification mapping,

$$\langle m \leftarrow l \rangle \mathbb{B}\mathfrak{s}(l) \subseteq \mathbb{B}\mathfrak{s}(m). \quad (30)$$

Similarly, $\langle l \leftarrow m \rangle \mathbb{B}\mathfrak{s}(m) \subseteq \mathbb{B}\mathfrak{s}(l)$. Applying to the last inclusion unification mapping $\langle m \leftarrow l \rangle$, and using properties 9.1 as well as correlation (28) we obtain the inverse inclusion to (30).

4) Note, that: $\bigcup_{\alpha \in \mathcal{A}} A_\alpha = \mathbb{B}\mathfrak{s}(l) \setminus (\bigcap_{\alpha \in \mathcal{A}} (\mathbb{B}\mathfrak{s}(l) \setminus A_\alpha))$. Hence, using items 1, 2 and 3 of this assertion we obtain:

$$\begin{aligned} \langle m \leftarrow l \rangle \left(\bigcup_{\alpha \in \mathcal{A}} A_\alpha \right) &= \langle m \leftarrow l \rangle \mathbb{B}\mathfrak{s}(l) \setminus \left(\bigcap_{\alpha \in \mathcal{A}} (\langle m \leftarrow l \rangle \mathbb{B}\mathfrak{s}(l) \setminus \langle m \leftarrow l \rangle A_\alpha) \right) = \\ &= \mathbb{B}\mathfrak{s}(m) \setminus \left(\bigcap_{\alpha \in \mathcal{A}} (\mathbb{B}\mathfrak{s}(m) \setminus \langle m \leftarrow l \rangle A_\alpha) \right) = \bigcup_{\alpha \in \mathcal{A}} \langle m \leftarrow l \rangle A_\alpha. \end{aligned}$$

5) Let $A \subseteq \mathbb{B}\mathfrak{s}(l)$, and $A = \{\omega\}$ is a singleton. By remark 11.4, $l \succ! m$ and, since $A \neq \emptyset$, by definition of normal visibility, we have $\langle m \leftarrow l \rangle A \neq \emptyset$. Suppose, that the set $B = \langle m \leftarrow l \rangle A$ contains more, than one element. Then, there exist sets $B_1, B_2 \subseteq B$ such, that $B_1, B_2 \neq \emptyset$ and $B = B_1 \sqcup B_2$. Denote: $A_1 := \langle l \leftarrow m \rangle B_1$, $A_2 := \langle l \leftarrow m \rangle B_2$. Since $B_1, B_2 \neq \emptyset$, then, by the definition of normal visibility, $A_1, A_2 \neq \emptyset$. Since $B = B_1 \sqcup B_2$, then, by the definition of precise visibility, $\langle l \leftarrow m \rangle B = \langle l \leftarrow m \rangle B_1 \sqcup \langle l \leftarrow m \rangle B_2 = A_1 \sqcup A_2$. Hence, taking into account, that $B = \langle m \leftarrow l \rangle A$ and using the equality (28), we obtain:

$$A_1 \sqcup A_2 = \langle l \leftarrow m \rangle B = \langle l \leftarrow m \rangle \langle m \leftarrow l \rangle A = A.$$

Thus, we see, that the set A can be divided into two nonempty disjoint sets, which contradicts the fact, that the set A is a singleton. Therefore, the set $\langle m \leftarrow l \rangle A$ is nonempty, and can not contain more, than one element, hence, it is a singleton. \square

Definition 11.9. Let \mathcal{Z} be a precisely visible changeable set, $l, m \in \mathcal{L}k(\mathcal{Z})$ and $\omega \in \mathbb{B}\mathfrak{s}(l)$. Elementary-time state $\omega' \in \mathbb{B}\mathfrak{s}(m)$ such, that $\{\omega'\} = \langle m \leftarrow l \rangle \{\omega\}$ will be referred to as **visible image** of elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(l)$ in the area of perception m and it will be denoted by $\langle ! m \leftarrow l \rangle \omega$:

$$\omega' = \langle ! m \leftarrow l \rangle \omega.$$

By the assertion 11.9, item 5, any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(l)$ always has a visible image $\omega' = \langle ! m \leftarrow l \rangle \omega$. Hence, by definition 11.9, for any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(l)$ in the area of perception $l \in \mathcal{L}k(\mathcal{Z})$ of precisely visible changeable set \mathcal{Z} the following equality holds:

$$\langle m \leftarrow l \rangle \{\omega\} = \{\langle ! m \leftarrow l \rangle \omega\} \quad (m \in \mathcal{L}k(\mathcal{Z})) \quad (31)$$

Using the equality $A = \bigsqcup_{\omega \in A} \{\omega\}$, definition of precise visibility and equality (31) we obtain the following theorem.

Theorem 11.2. *For any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ in area of perception $l \in \mathcal{L}k(\mathcal{Z})$ of precisely visible changeable set \mathcal{Z} the following equality is true:*

$$\langle m \leftarrow l \rangle A = \bigsqcup_{\omega \in A} \{ \langle ! m \leftarrow l \rangle \omega \} = \{ \langle ! m \leftarrow l \rangle \omega \mid \omega \in A \} \quad (m \in \mathcal{L}k(\mathcal{Z})). \quad (32)$$

Corollary 11.2. *Let \mathcal{Z} be a precisely visible changeable set and $l, m \in \mathcal{L}k(\mathcal{Z})$ any it's areas of perception.*

Then for any changeable system $A \subseteq \mathbb{B}\mathfrak{s}(l)$ the sets A and $\langle m \leftarrow l \rangle A$ are equipotent, in particular the sets $\mathbb{B}\mathfrak{s}(l)$ and $\mathbb{B}\mathfrak{s}(m)$ are equipotent. In the case $A \neq \emptyset$ the mapping:

$$f(\omega) = \langle ! m \leftarrow l \rangle \omega, \quad \omega \in \mathbb{B}\mathfrak{s}(l) \quad (33)$$

is bijection between the sets A and $\langle m \leftarrow l \rangle A$.

Proof. In the case $A = \emptyset$ the statement of the corollary follows from the assertion 9.3. In the case $A \neq \emptyset$ from the theorem 11.2 (equality (32)) it follows, that the mapping (33) is bijection between the sets A and $\langle m \leftarrow l \rangle A$. And from the assertion 11.9 (item 3)) it follows, that the sets $\mathbb{B}\mathfrak{s}(l)$ and $\mathbb{B}\mathfrak{s}(m)$ are equipotent. \square

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